# Lecture 16 (2/14)

Monday, February 12, 2024 12:12 PM

**Thm** There is a bijection

$$\left\{ \begin{matrix} \text{regular maps} \\ X \xrightarrow{f} \mathbb{P}^n \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{line bdles } E \to X \\ + \text{ sections } s_0, \dots, s_n \\ \text{such that } s_i \neq 0 \text{ simultaneously} \end{matrix} \right\}$$

**Pf** (1) Suppose we are given a line bdle $E$ sections $s_0, \dots, s_n$, define

$$f(x) = [s_0(x) : \dots : s_n(x)]$$

What does it mean? Choose a basis $e$ in $\pi^{-1}(x)$

$$s_i(x) = \alpha_i \cdot e, \text{ so } [s_0(x) : \dots : s_n(x)] = [\alpha_0 : \dots : \alpha_n]$$
$$\alpha_i \in \mathbb{K}$$

If we rescale basis $e \to \lambda e$, then $\alpha_i \to \frac{\alpha_i}{\lambda}$, but $[\alpha_0 : \dots : \alpha_n] = [\frac{\alpha_0}{\lambda} : \dots : \frac{\alpha_n}{\lambda}]$ in $\mathbb{P}^n$, and the map is well defined.

(2.) Given a map $f: X \to \mathbb{P}^n$, we can define $E = f^* \mathcal{O}(1)$, and $s_i = f^* x_i$.

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**Rmk** $E$ is called generated by sections if there exist sections $s_0, \dots, s_n$ which do not vanish simultaneously.

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Exa. 1) $X = \mathbb{P}^1$ $E = \mathcal{O}(1)$ sections = deg 1 poly... vanish

**Examples** 1) $X = \mathbb{P}^1_{[x_0:x_1]}$, $E = \mathcal{O}(1)$ sections = deg 1 polynomials.

$s_0, \dots, s_n \qquad s_i = \alpha_i x_0 + \beta_i x_1$

$$f([x_0 : x_1]) = [\alpha_0 x_0 + \beta_0 x_1 : \dots : \alpha_n x_0 + \beta_n x_1]$$

$$= x_0[\alpha_0 : \dots : \alpha_n] + x_1[\beta_0 : \dots : \beta_n]$$

$f : \mathbb{P}^1 \longrightarrow \mathbb{P}^n$ image = line in $\mathbb{P}^n$

2) $X = \mathbb{P}^1_{[x_0:x_1]}$, $E = \mathcal{O}(k)$

$$f([x_0 : x_1]) = [x_0^k : x_0^{k-1} x_1 : \dots : x_1^k]$$

$f : \mathbb{P}^1 \longrightarrow \mathbb{P}^k$ image = "Veronese curve"

(twisted cubic for $n=3$).

**Exercise:** find the equations for the image of $f$ for all $k$.

3) $X = \mathbb{P}^1_{[x_0:x_1]}$, $E = \mathcal{O}(3)$

$f([x_0 : x_1]) = [s_0 : \dots : s_n]$ $\quad s_i = a_i x_0^3 + b_i x_0^2 x_1 + c_i x_0 x_1^2 + d_i x_1^3$

some cubic polynomials

All such morphisms factor as compositions

$$\mathbb{P}^1 \xrightarrow[\text{twisted cubic}]{} \mathbb{P}^3 \xrightarrow{\phi} \mathbb{P}^n$$

$[x_0 : x_1] \longrightarrow [\underset{y_0}{x_0^3} : \underset{y_1}{x_0^2 x_1} : \underset{y_2}{x_0 x_1^2} : \underset{y_3}{x_1^3}]$ $\qquad \phi([y_0 : y_1 : y_2 : y_3]) =$

$$= [a_i y_0 + b_i y_1 + c_i y_2 + d_i y_3]$$

$y_0 \quad y_1 \quad y_2 \quad y_3 \qquad = (a_i y_0 + b_i y_1 + c_i y_2 + d_i y_3)$

4) $X = \mathbb{P}^n \times \mathbb{P}^m \qquad \mathcal{O}(k,l) = p_1^* \mathcal{O}(k) \otimes p_2^* \mathcal{O}(l)$

$$\begin{tikzcd} & \mathbb{P}^n \times \mathbb{P}^m \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ \mathbb{P}^n & & \mathbb{P}^m \end{tikzcd}$$

$k, l \in \mathbb{Z}$

$\mathbb{Z}^2$ worth of line bundles on $\mathbb{P}^n \times \mathbb{P}^m$

Sections of $\mathcal{O}(k,l)$ = polynomials in $x_i, y_j$ homog. of degree $k$ in $x_i$ and of deg $l$ in $y_j$

<u>Ex:</u> $\mathcal{O}(1,1)$ is generated by sections $x_i y_j$

$f : \mathbb{P}^n \times \mathbb{P}^m \longrightarrow \mathbb{P}^N$ is the Segre embedding

$N = (m+1)(n+1) - 1.$

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## Graded coordinate rings

$$\mathbb{K}[x_0 \ldots x_n] = \bigoplus_{k=0}^{\infty} \left\{ \begin{array}{c} \text{degree } k \\ \text{polynomials} \end{array} \right\} = \bigoplus_{k=0}^{\infty} \Gamma(\mathcal{O}(k))$$

$\mathbb{K}[x_0 \ldots x_n]$ = homogeneous coordinate ring of $\mathbb{P}^n$

<u>Product:</u> $\mathcal{O}(k) \otimes \mathcal{O}(l) \xrightarrow{\sim} \mathcal{O}(k+l)$

$$\Gamma(\mathcal{O}(k)) \otimes \Gamma(\mathcal{O}(l)) \longrightarrow \Gamma(\mathcal{O}(k+l))$$

More generally, if $E \to X$ is a line bundle, we can consider

$$A = \bigoplus_{k=0}^{\infty} \Gamma(E^{\otimes k}) = \underbrace{\Gamma(\mathcal{O}_X)}_{\text{global functions}} \oplus \Gamma(E) \oplus \Gamma(E \otimes E) \oplus \ldots$$

Product: $\Gamma(E^{\otimes k}) \otimes \Gamma(E^{\otimes l}) \to \Gamma(E^{\otimes k+l})$

functions

Product: $\Gamma(E^{\otimes k}) \otimes \Gamma(E^{\otimes l}) \longrightarrow \Gamma(E^{\otimes k+l})$

Ex: $\Gamma(E) \otimes \Gamma(E) \longrightarrow \Gamma(E \otimes E)$

$s_1 \quad s_2 \quad s_1 s_2$

If $s_0, \ldots, s_n$ are sections of $E$ then we get a homomorphism $\mathbb{K}[x_0 : \ldots : x_n] \longrightarrow A$

$g(x_0, \ldots, x_n) \longrightarrow g(s_0, \ldots, s_n)$

homogeneous degree $k$

$g(s_0, \ldots, s_n) \in \Gamma(E^{\otimes k})$