Dimension

X = alg. variety

Def $\dim X$ = length of maximal chain of irreducible closed

$$\emptyset \neq Z_0 \subsetneq Z_1 \subsetneq Z_2 \subsetneq \dots \subsetneq Z_n$$ subsets of X.

Easy: $\dim \mathbb{A}^1 = 1$ (over $\mathbb{K} = \overline{\mathbb{K}}$). Indeed, any closed subsets of $\mathbb{A}^1$ are $\emptyset$, $\mathbb{A}^1$ and finite sets. So max chain is $\{pt\} \subset \mathbb{A}^1$.

Hard $\dim \mathbb{A}^n = n$. Proof below; will need many commutative algebra.

Lemma $X = X_1 \cup \dots \cup X_k$ irr. components. Then $\dim X = \max(\dim X_i)$

Pf: Recall $Z \subset X$ closed $\Rightarrow Z = (Z \cap X_1) \cup \dots \cup (Z \cap X_k)$

If Z is irreducible then $Z = Z \cap X_i$ for some i, so $Z \subset X_i$. Therefore for a chain

$Z_1 \subset \dots \subset Z_n$ we get $Z_n \subset X_i$ for some i

and $n \leq \dim X_i$, so $\dim X \leq \max(\dim X_i)$.

Conversely, by def. $\dim X \geq \dim X_i$. $\blacksquare$

Def A = algebra, then (Krull) dimension of A is defined as the length of a maximal chain is defined as the length of a maximal chain of prime ideals in A: $p_1 \subsetneq p_2 \subsetneq p_3 \subset \cdots \subsetneq p_n \subset A$

Def A = algebra over $\mathbb{K}$, $a_1 \ldots a_n$ are called algebraically independent if $f(a_1 \ldots a_n) = 0$ for a polynomial f implies $f = 0$.

The trancendence degree of A is

$$\operatorname{trdeg}(A) = \max\{n : \exists\, a_1 \ldots a_n \text{ algebraically independent in } A\}$$

Note: $\operatorname{trdeg}(A) \geq n \iff$ there exist $a_1 \ldots a_n$ alg. independent $\iff$ there exists an injective homomorphism $\mathbb{K}[x_1 \ldots x_n] \longrightarrow A$. $(x_i \longrightarrow a_i)$.

Note: If $A = A(X)$ for affine alg. set X, then injective homomorphism $\mathbb{K}[x_1 \ldots x_n] \longrightarrow A(X)$ $\iff$ dominant map $X \longrightarrow \mathbb{A}^n$.

Cor $\operatorname{trdeg}(A(X)) \geq n \iff$ there exists a dominant map $X \longrightarrow \mathbb{A}^n$.

Lemma1 Assume A is an integral domain (no zero divisors) and $I \subset A$ nonzero ideal. Then

$\operatorname{trdeg}(A/I) < \operatorname{trdeg} A$.

__Proof__ Assume $\operatorname{trdeg} A = n$, we need to prove that any n elts in A/I are algebraically dependent.

Choose $a_1 \ldots a_n \in A$ and $a \in I$, then $a_1 \ldots a_n, a$ are alg. dependent and $g(a_1 \ldots a_n, a) = 0$

Write $\sum a^i g_i(a_1 \ldots a_n) = 0$.

If $g_0 = 0$, we can divide by a (since A is a domain), so we can assume $g_0 \neq 0$ and

$$g_0(a_1 \ldots a_n) = -\sum_{i>0} a^i g_i(a_1 - a_n) \in I$$

So $g_0(\overline{a_1}, \ldots \overline{a_n}) = 0$ in A/I. ☒

__Lemma 2__ Assume A is a domain, and $\operatorname{trdeg}(A) = 0$. Then A is a field.

__Proof__ similar to lemma 1,

we can write $\sum a^i g_i = 0$ and $g_0 \neq 0$.

Then $1 = \frac{1}{g_0} a \cdot \left(-\sum_{i \geq 1} g_i a^{i-1}\right)$, hence a is invertible and A is a field. ☒

__Thm__ For any algebra A we have

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$$\dim A \leq \operatorname{trdeg} A.$$

Proof: Suppose that $\operatorname{trdeg} A = n$, we run induction on n

Base case: $n=0$ If $\operatorname{trdeg} A = 0$ and $p_0 \subset A$ prime then A/p_0 is a domain with $\operatorname{trdeg}(A/p_0) = 0$, so by lemma 2 A/p_0 is a field. Therefore there are no ideals in A/p_0, and $\dim A \leq 0$.

Step of induction Assume we proved the statement for $\operatorname{trdeg} \leq n-1$

Suppose we have a chain of prime ideals

$$p_0 \subsetneq p_2 \subsetneq \cdots \subsetneq p_k \subset A.$$

Then A/p_0 is a domain and we get a chain

$$0 \subsetneq p_1/p_0 \subsetneq p_2/p_0 \subsetneq \cdots \subsetneq p_k/p_0 \subset A/p_0$$

By lemma 1, $\operatorname{trdeg}(A/p_1) = \operatorname{trdeg}\left(\frac{A/p_0}{p_1/p_0}\right) < \operatorname{trdeg}(A/p_0) = n$

Therefore $k-1 \leq \dim(A/p_1) \leq \operatorname{trdeg}(A/p_1) < n$

by assumption of induction

and $k \leq n$.

Thm $\operatorname{trdeg}(\mathbb{k}[x_1, \ldots, x_n]) = n$

Thm $\operatorname{trdeg}(\mathbb{K}(x_1, \dots x_n)) = n$

Proof: $x_1 \dots x_n$ clearly algebraically independent.
We need to prove that any $(n+1)$ polynomials $f_1 \dots f_{n+1}$ are dependent. Assume $d = \max(\deg f_i)$. There are $\sim \frac{k^{n+1}}{(n+1)!}$ products $f_1^{\alpha_1} \dots f_{n+1}^{\alpha_{n+1}}$, $\alpha_i \leq k$ all of degree $< dk$. On the other hand, there are $\sim \frac{(dk)^n}{n!}$ monomials in x_i of degree $\leq dk$.

For large enough k, $\frac{k^{n+1}}{(n+1)!} \gg \frac{(dk)^n}{n!}$ and the result follows. $\boxtimes$

Thm $\dim \mathbb{A}^n = \dim \mathbb{K}[x_1 \dots x_n] = n$.

Proof: By Thm, $\dim \mathbb{K}[x_1 \dots x_n] \leq \operatorname{trdeg} \mathbb{K}[x_1 \dots x_n] = n$.
On the other hand, $(0) \subset (x_1) \subset (x_1, x_2) \subset \dots \subset (x_1 \dots x_n)$ is a chain of prime ideals corresponding to $\{pt\} \subset \mathbb{A}^1 \subset \dots \subset \mathbb{A}^n$.
So $\dim \mathbb{K}[x_1 \dots x_n] \geq n$. $\boxtimes$
