Lecture 18 (2/21)

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Recap:

(1) $\dim A \le \operatorname{trdeg} A$

(2) $\operatorname{trdeg}(A/I) < \operatorname{trdeg} A$ for $I \neq 0$

(3) $\operatorname{trdeg} \mathbb{K}[x_1 .. x_n] = n$.

Cor $\dim \mathbb{A}^n = \dim \mathbb{K}[x_1 - x_n] = n$.

Proof By (3) $\dim \mathbb{A}^n \underset{(1)}{\le} n$, but we can find a chain $0 < (x_1) \subset (x_1, x_2) \subset -- \subset (x_1, ... x_n)$ of length n

$\Rightarrow \dim \mathbb{A}^n \geq n$. ∎

Cor $A(X) = \dfrac{\mathbb{K}[x_- x_n]}{I}$, then $\dim X = \dim A(X) \le n$.

Proof If $I = 0$ then $X = \mathbb{A}^n$ and $\dim = n$.

Otherwise $\dim A(x) \underset{(1)}{\le} \operatorname{trdeg} \dfrac{\mathbb{K}(x_1 - x_n]}{I} \underset{(2)}{<} \operatorname{trdeg} \mathbb{K}(x_1 - x_n] = n$

∎

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lemma $X = X_1 \cup -- \cup X_k$

irreducible components

Then $\operatorname{trdeg} A(X) \le \max \operatorname{trdeg} A(X_i)$.

Pf: Assume $a_1 .. a_n \in A(X)$ alg. independent, $\underset{n}{\overset{\operatorname{trdeg} A(x)}{\shortparallel}}$.

but $a_1|_{X_i} ... a_n|_{X_i}$ are alg. dependent for all i.

Then $f_i(a_1 ... a_n) = 0$ on X_i

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$\Rightarrow \prod f_i(a_1 \dots a_n) = 0$ on X

$\Rightarrow a_1 \dots a_n$ are alg. dependent in $A(X)$, contradiction.

Therefore $\operatorname{trdeg} A(X_i) \geq n$ for some i.

<u>Thm</u> Suppose X = affine alg. set in $\mathbb{A}^N$

Then $\dim X = \dim A(X) = \operatorname{trdeg} A(X)$.

<u>Pf</u> ① Assume X is irreducible, then $A(X)$ = integral domain

Suppose $\operatorname{trdeg} A(X) = n$, we already know $\dim A(X) \leq n$;

and need to prove $\dim A(X) \geq n$.

That is, we need to construct a

chain of prime ideals, by induction on n.

$a_1 \dots a_n$ = alg. independent, define $L = K(a_1)$ = field

$a_2 \dots a_n$ are alg. independent over $L \Rightarrow \operatorname{trdeg}_L A(X) \geq n-1$

Define $A' = L \cdot A(X) \subset \operatorname{Quot}(A)$.

$\Rightarrow$ by assumption of induction we can find

$$p_0' \subsetneq p_1' \subsetneq \dots \subsetneq p_{n-1}' \subsetneq A'$$

Define $p_i = p_i' \cap A$. We need to check:

- p_i are prime. Indeed, if $f, g \in A$, $fg \in p_i$

- p_i are prime. Indeed, if $f, g \in A$, $fg \in p_i$
  then $fg \in p_i' \Rightarrow f \in p_i'$ or $g \in p_i' \Rightarrow f \in p_i$ or $g \in p_i$
- $p_i \neq p_{i+1}$. Indeed, $p_i' = p_i \cdot L \Rightarrow$ if $p_i = p_{i+1}$, then
  $p_i' = p_{i+1}'$, contradiction
- p_{n-1} is not maximal. Indeed, assume it is, then A/p_{n-1} is a field. Then a_1 is algebraic(*) in A/p_{n-1}, so that $f(a_1) \in p_{n-1}$. Therefore

  $f(a_1) \in p_{n-1} \cap L$, so invertible over L, and

  $p_{n-1}' = A$, contradiction.
- We conclude that p_{n-1} is not maximal, and we can take p_n = maximal ideal containing p_{n-1}.

We get a chain $p_0 \subsetneq p_1 \subsetneq \cdots \subsetneq p_{n-1} \subsetneq p_n$.

So $\dim A(X) \geq n$ ∎

(*) Here we used fact, see <u>Lecture 3</u>.

$\boxed{\text{Fact}}$ If $K \subset M$ is a finitely generated field extension then any element of M is algebraic over K.

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(2) Suppose X is reducible, $X = X_1 \cup \cdots \cup X_k$ ← irr. components.

$\dim X = \max(\dim X_i) =$

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$= \max(\operatorname{trdeg} A(X_i)) \geq \operatorname{trdeg} A(X)$ by lemma.

On the other hand, $\dim X \leq \operatorname{trdeg} A(X)$, so they are equal. ∎

Cor Suppose $\varphi : X \to Y$ dominant, then

$\dim X \geq \dim Y$.

Proof: φ is dominant $\iff \varphi^* : A(Y) \to A(X)$ injective.

If $a_1 \ldots a_n \in A(Y)$ alg. independent, then

$\varphi^*(a_1), \ldots \varphi^*(a_n)$ are alg. independent

$\big(f(\varphi^*(a_1), \ldots, \varphi^*(a_n)) = \varphi^*(f(a_1, \ldots a_n)) = 0$

implies $f(a_1 \ldots a_n) = 0\big)$

$\Rightarrow \operatorname{trdeg} A(X) \geq n = \operatorname{trdeg} A(Y)$. ∎

Cor X irreducible, proper closed subset of Y.

Then $\dim X < \dim Y$.

Proof I $Y = Y_1 \cup \ldots \cup Y_k$ — irred. components,

$X \subset Y_i$ for some i, then any chain

$X_0 \subsetneq X_1 \subseteq \ldots \subseteq X_n \subset X$ extends to

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$X_0 \subsetneq X_1 \subseteq \dots \subseteq X_n \subseteq Y_i$,

so $\dim Y \geq n+1$.

Proof II $A(X) = A(Y)/I$, $\operatorname{tr\,deg} A(X) < \operatorname{tr\,deg} A(Y)$. $\blacksquare$
