Lecture 19 (2/21)

Wednesday, February 21, 2024 2:03 PM

Properties of dimension

Recap:

1. X = affine algebraic set, then $\dim X = \operatorname{trdeg} A(X)$
2. $X \subseteq Y \Rightarrow \dim X \leq \dim Y$
3. $X \underset{\neq Y}{}$ closed in Y, Y irreducible $\Rightarrow \dim X < \dim Y$
4. $U \subseteq X$ open (nonempty), X irreducible $\Rightarrow \dim U = \dim X$

Proof There is $D(f) = \{f \neq 0\}$ such that

$$D(f) \subseteq U \subseteq X \qquad \underline{D(f) \text{ affine}}$$

$$\Rightarrow \underset{\operatorname{trdeg} A(X)[f^{-1}]}{\underset{\|}{\dim D(f)}} \leq \dim U \leq \underset{\operatorname{trdeg} A(X)}{\underset{\|}{\dim X}}$$

Fact (w/o proof) If A is an integral domain then $\operatorname{trdeg} A = \operatorname{trdeg} \operatorname{Quot}(A) \leftarrow$ field of fractions

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Therefore $\operatorname{trdeg} A(X)[f^{-1}] = \underset{\operatorname{trdeg} \operatorname{Quot}(A)}{\underset{\|}{\operatorname{trdeg} A(X)}} = \dim X$

and we are done. ∎

Cor U = open subset in $\mathbb{A}^n$ (nonempty)

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$\Rightarrow \dim \mathcal{U} = n$.

(5) $\dim(X \times Y) = \dim X + \dim Y$ (w/o proof)

(6) Thm (Krull principal ideal theorem)

X irreducible, $f \in A(X)$ and $\{f=0\} \neq X, \emptyset$.

Then all irreducible components of $\{f=0\}$ have dimension exactly $\dim X - 1$.

Rmk $\operatorname{trdeg} A/(f) = \operatorname{trdeg} A - 1$.

Proof: Skip, see e.g. Clader-Ross section 6.6.

Cor Suppose X irreducible, and

$Y = \{f_1 = \dots = f_k\}$ in X.

Then all irreducible components of Y have dimension $\dim(Y_i) \geq \dim X - k$.

Proof Induction on k. Suppose

$Z = \{f_1 = \dots = f_{k-1}\}$, then by assumption of induction all irreducible components of Z have dimension $\dim Z_i \geq \dim X - k + 1$

Now $Y = \{f_k = 0\} \cap Z = \bigcup(\{f_k = 0\} \cap Z_i)$

By Thm, we have the following cases:

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- $\{f_k=0\} \cap Z_i = \emptyset$ OK
- $\{f_k=0\} \cap Z_i = Z_i \Rightarrow \dim \geq \dim X - k + 1$
- $\{f_k=0\} \cap Z_i \neq Z_i, \emptyset \Rightarrow$ by Thm all irred. components of $\{f_k=0\} \cap Z_i$ have $\dim \geq \dim X - k$. ☒

<u>Def</u> $Y = \{f_1 \dots f_k\} \subset X$ is called a <u>complete intersection</u> in X if for all irreducible components we have $\dim Y_i = \dim X - k$.

<u>Example</u>: twisted cubic

$$\mathbb{P}^1 \xrightarrow{\varphi} \mathbb{P}^3 \qquad C = \varphi(\mathbb{P}^1)$$

$$[x_0 : x_1] \longrightarrow [y_0 : y_1 : y_2 : y_3] = [x_0^3 : x_0^2 x_1 : x_0 x_1^2 : x_1^3]$$

HW: C closed, defined by 3 equations

$$y_0 y_2 = y_1^2, \quad y_1 y_3 = y_2^2, \quad y_0 y_3 = y_1 y_2.$$

Since $\mathbb{P}^1 \underset{\varphi}{\cong} C$, $\dim \mathbb{P}^1 = \dim C = 1$.

By thm, $\dim C \geq \dim \mathbb{P}^3 - \#\text{equations} =$

$$= 3 - 3 = 0.$$

This is fine, but can we see what's going on?

- $y_0 y_3 = y_1 y_2$ is nondegenerate quadric. Q

- $y_0 y_3 = y_1 y_2$ is nondegenerate quadric Q, irreducible, $\dim = 3-1 = 2$.

(in fact, it is Segre quadric $\simeq \mathbb{P}^1 \times \mathbb{P}^1$).

- $\{y_0 y_3 = y_1 y_2,\ y_0 y_2 = y_1^2\} = Z$

By thm we know that Z is defined by one equation in Q, so <u>all</u> components of Z have dimension $\dim Q - 1 = 2-1 = 1$.

What are they?

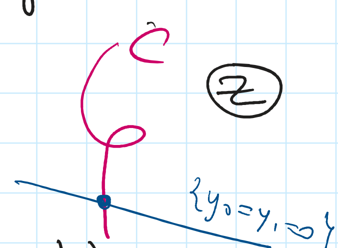
1) If $y_0 = 0$ then from $y_0 y_2 = y_1^2$ we get $y_1 = 0$ and the first equation holds automatically.

So we get a line $\{y_0 = y_1 = 0\}$

2) If $y_0 \neq 0$ then we can assume $y_0 = 1$ and $y_2 = y_1^2$, $y_3 = y_1 y_2 = y_1^3$, so

$[y_0 : y_1 : y_2 : y_3] = [1 : y_1 : y_1^2 : y_1^3]$.

This is our twisted cubic C (in chart).



So $Z = C \cup \{y_0 = y_1 = 0\}$ has 2 irreducible components, both $\dim = 1$.

Intersect at $[0:0:0:1]$

- Finally, we write the last equation $\{y_1 y_3 = y_2^2\}$.

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$y_2 = 0$, so $[y_0 : y_1 : y_2 : y_3] = [0:0:0:y_3] = [0:0:0:1]$.

Thm $X, Y \subset \mathbb{A}^n$ irreducible, closed.
Then all components of $X \cap Y$ have dimension
at least $\dim X + \dim Y - n$.

Pf: Consider $X \times Y \subset \underset{x_1 - x_n}{\mathbb{A}^n} \times \underset{y_1 - y_n}{\mathbb{A}^n} = \mathbb{A}^{2n}$ $\qquad \dim X \times Y = \dim X + \dim Y$

Let $Z = X \times Y \cap \{x_1 = y_1, \ldots x_n = y_n\}$

On the one hand, $Z \simeq X \cap Y$, on the other

Z is cut out by n equations in $X \times Y$

$\Rightarrow \dim Z \geq \dim X \times Y - n = \dim X + \dim Y - n.$ $\boxtimes$
