Lecture 2 (1/10)

Tuesday, January 9, 2024 5:11 PM

Lemma I_1, I_2 = ideals

(a) $Z(I_1+I_2) = Z(I_1)\cap Z(I_2)$

(b) $Z(I_1\cap I_2) = Z(I_1)\cup Z(I_2)$

(c) $Z(1)=\emptyset$ (d) $Z(0)=\mathbb{A}^n$

Pf (a), (c), (d) clear by defn. since I_1+I_2 = ideal gen. by $I_1\cup I_2$.

(b): Assume $p\in Z(I_1)\cup Z(I_2)$.

Then either $p\in Z(I_1)$ or $p\in Z(I_2)$

Given $f\in I_1\cap I_2$, f belongs both to p_1 and p_2

$\Rightarrow f(p)=0$. So $Z(I_1)\cup Z(I_2)\subset Z(I_1\cap I_2)$.

Conversely, assume $p\in Z(I_1\cap I_2)$ but $p\notin Z(I_1)$, $p\notin Z(I_2)$.

Then there exist $f\in I_1$, $g\in I_2$ such that

$f(p)\neq 0$, $g(p)\neq 0$. Then $f(p)g(p)\neq 0$.

On the other hand $fg\in I_1\cap I_2$ since I_1, I_2 are ideals.

Contradiction. $\boxtimes$

Def I = ideal in $A=\mathbb{K}(x_1 \dots x_n)$

$\sqrt{I} = \{f : f^N\in I \text{ for some } N\}$ = radical of I

I is called <u>radical</u> ideal if $\sqrt{I}=I$.

Lemma a) $\sqrt{I}$ is an ideal in A

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b) $\sqrt{\sqrt{I}} = \sqrt{I}$, so $\sqrt{I}$ is radical

c) $Z(\sqrt{I}) = Z(I)$.

Proof a) • Suppose $f^N \in I$ and g arbitrary

Then $(fg)^N = f^N \cdot g^N \in I \Rightarrow fg \in \sqrt{I}$

• Suppose $f^{N_1} \in I$, $g^{N_2} \in I$ then

$$(f+g)^{N_1+N_2} = \sum_{a+b=N_1+N_2} \binom{N_1+N_2}{a} f^a g^b$$

In all terms either $a \geq N_1$ or $b \geq N_2 \Rightarrow f^a g^b \in I$

$\Rightarrow (f+g)^{N_1+N_2} \in I \Rightarrow f+g \in \sqrt{I}$.

b) Clear / exercise.

c) $I \subset \sqrt{I} \Rightarrow Z(I) \supset Z(\sqrt{I})$.

Conversely, suppose $p \in Z(I)$ and $f \in \sqrt{I}$.

Then $f^N \in I \Rightarrow f^N(p) = 0 \Rightarrow f(p) = 0$

So $p \in Z(\sqrt{I})$. $\blacksquare$

Ex $I = (x^3, y^7)$ then $\sqrt{I} = (x, y)$

$Z(I) = \{x^3 = y^7 = 0\} = \{(0,0)\} = \{x = y = 0\}$.

Thm 1 (Hilbert's Nullstellensatz, part 1)

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Suppose $\mathbb{K}$ is algebraically closed. Then all maximal ideals in $\mathbb{K}[x_1 .. x_n]$ are of the form

$(x_1 - a_1, x_2 - a_2, ... x_n - a_n)$ for some $(a_1 .. a_n) \in \mathbb{A}^n$.

Proof: Friday lecture.

Thm 2 (Nullstellensatz, p.2) Suppose $\mathbb{K}$ is algebraically closed and $I \subset \mathbb{K}[x_1 .. x_n]$ is a proper ideal. Then $Z(I) \neq \emptyset$.

Pf I is contained in some maximal ideal m.
By Thm 1, $m = (x_1 - a_1, ... x_n - a_n)$ for some $(a_1 .. a_n)$

$\underset{(a_1 .. a_n)}{\underset{\parallel}{Z(m)}} \subset Z(I)$, so $Z(I)$ contains the point $(a_1 .. a_n)$.

∎

Thm 3 (Nullstellensatz, part 3) Suppose $\mathbb{K}$ is alg. closed, I = some ideal, and f vanishes at all points of $Z(I)$. Then $f^N \in I$ for some N, so that $f \in \sqrt{I}$.

Proof Add a variable x_{n+1}, consider new ideal

$J = I + (x_{n+1} f(x_1 .. x_n) - 1)$

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What is $Z(J)$? If $(x_1 \ldots x_n, x_{n+1}) \in Z(J)$

then
- $g(x_1 \ldots x_n) = 0$ for all $g \in I$

$$\Rightarrow (x_1, \ldots, x_n) \in Z(I)$$

- $x_{n+1} f(x_1 \ldots x_n) = 1$. But by our assumption $f(x_1 \ldots x_n) = 0$, contradiction.

Therefore $Z(J) = \emptyset$ and by Thm 2

$J = \mathbb{K}[x_1 \ldots x_n, x_{n+1}]$. In particular, $1 \in J$.

$$1 = g + h \cdot (x_{n+1} f(x_1 \ldots x_n) - 1)$$

$g \in I[x_{n+1}]$, h arbitrary.

Let $y = \frac{1}{x_{n+1}}$, $\quad g(x_1 \ldots x_n, x_{n+1}) = g(x_1 \ldots x_n, \frac{1}{y}) =$

can choose same

$$= \frac{1}{y^N} \tilde{g}(x_1 \ldots x_n, y)$$

$$h(x_1 \ldots x_n, x_{n+1}) = \frac{1}{y^{N-1}} \tilde{h}(x_1 \ldots x_n, y)$$

$$1 = \frac{1}{y^N} \left[\tilde{g} + \tilde{h} \cdot (f(x_1 \ldots x_n) - y) \right]$$

$$y^N = \tilde{g} + \tilde{h}(f(x_1 \ldots x_n) - y) \qquad \tilde{g} \in I$$

Now plug in $y = f(x_1 \ldots x_n)$

$$\boxed{f(x_1 \ldots x_n)^N = \tilde{g} \in I}.$$

$\boxed{f'(x_1 \dots x_n)^N = g \in I}$.