<u>Thm</u> Let $m_p = (x_1 - p_1, \dots, x_n - p_n) \subset A(X)$ be the maximal ideal in $A(X)$ corresponding to p. Then $(T_p X)^* = m_p / m_p^2$.

<u>Pf</u>: ① Given $b = (b_1, \dots, b_n) \in T_p X$, we define a linear functional $m_p \xrightarrow{l_b} \mathbb{K}$

$$l_b(f) = L_p(f; b_1, \dots, b_n).$$

Note that $m_p = \dfrac{(x_1 - p_1, \dots, x_n - p_n)}{I(X)} \subset \dfrac{\mathbb{K}[x_1, \dots, x_n]}{I(X)} = A(X)$

and $l_b(f) = 0$ for $f \in I(X)$, so l_b is well defined on m_p.

Furthermore, $l_b(fg) = 0$ if $f, g \in m_p$, so

$l_b(m_p^2) = 0 \implies l_b : m_p / m_p^2 \longrightarrow \mathbb{K}$ is well defined.

② By ① we get a map $T_p X \longrightarrow (m_p / m_p^2)^*$

We need to prove it is an isomorphism. $X = \{f_1 = \dots = f_k = 0\}$

WLOG we can assume $p = (0, \dots, 0)$, then

$$m_p / m_p^2 = \frac{(x_1, \dots, x_n)}{(f_1, \dots, f_k) + (x_1, \dots, x_n)^2} = \frac{\mathrm{Span}(x_1, \dots, x_n)}{(\text{linear part of } f_i)}.$$

while $T_p X = \langle b \in \mathrm{Span}(x_1, \dots, x_n)^* : b(\text{linear part of } f_i = 0) \rangle$

while $T_pX = \langle b \in \mathrm{Span}(x_1 \dots x_n)^* : b(\text{linear part of } f_i = 0)\rangle$

And the result follows. ☒

Cor If $X \underset{\phi}{\xrightarrow{\sim}} Y$ then $T_pX \simeq T_{\phi(p)}Y$.

Proof $\phi^* : A(Y) \longrightarrow A(X)$ is a ring isomorphism

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Lemma T_pX is local, that is, if $U \subset X$ is open, $p \in U$ then $T_pU = T_pX$

Proof It is sufficient to consider $U = D(g) = \{g \neq 0\}$

Assume $g(p) \neq 0$, $f(p) = 0$ then

$$L_p\left(\frac{f}{g}; b\right) = \frac{g(p)L_p(f;b) - \overset{=0}{f(p)}L_p(g,b)}{g^2_{(p)}}$$

$$= \frac{1}{g(p)} L_p(f;b)$$

So $L_p\left(\frac{f}{g}; b\right) = 0$ iff $L_p(f;b) = 0$. ☒

Thm Suppose X is irreducible affine alg. set, then $\dim T_pX \geq \dim X$

Lemma $\dim T_pX = 0$ iff $\dim X = 0$.

Proof: • If $\dim X = 0$ and X is irreducible then X = point.

• Assume $\dim T_pX = 0$, then $m_p/m_p^2 = 0$.

therefore $m_p = m_p^2$

$\ldots \mathfrak{m}_p/\mathfrak{m}_p^2 = 0$.

therefore $\mathfrak{m}_p = \mathfrak{m}_p^2$.

Let $F_1 \ldots F_k$ = generators of $\mathfrak{m}_p$,

we can write $F_i = \sum G_{ij} F_j$ for $G_{ij} \in \mathfrak{m}_p$

then
$$(I - G)\begin{pmatrix} F_1 \\ \vdots \\ F_k \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \qquad (*)$$

Since $G_{ij}(p) = 0$, then $\det(I - G(p)) = 1$

$\Rightarrow U = \{\det(I - G(\overline{x})) \neq 0\}$ nonempty open containing p.

By $(*)$ we get $F_1 = \ldots = F_k = 0$ on U, contradiction ▨.

<u>Rmk</u> This is a variant of "Nakayama's lemma":

if R is a <u>local</u> ring with maximal ideal $\mathfrak{m}$

and $H_1 \ldots H_s \in \mathfrak{m}$ are such that $\overline{H}_1, \ldots \overline{H}_s$ span $\mathfrak{m}/\mathfrak{m}^2$

then $H_1 \ldots H_s$ generate $\mathfrak{m}$ as a ring.

<u>Proof</u>: Let $\tilde{R} = R/(H_1 \ldots H_s)$, then $\mathfrak{m}_{\tilde{R}} = \mathfrak{m}_R/(H_1 \ldots H_s)$

and $\mathfrak{m}_{\tilde{R}}/\mathfrak{m}_{\tilde{R}}^2 = 0$. Similar to Lemma we get $\mathfrak{m}_{\tilde{R}} = 0$

$\Rightarrow \mathfrak{m}_R = (H_1 \ldots H_s)$.

▨

<u>Proof of Thm</u>: induction on $\dim(X) = n$, $X \subset \mathbb{A}^N$ affine

① $n = 0 \Rightarrow \dim T_p X \geq 0$ OK

① $n=0 \Rightarrow \dim T_pX \geq 0$ OK

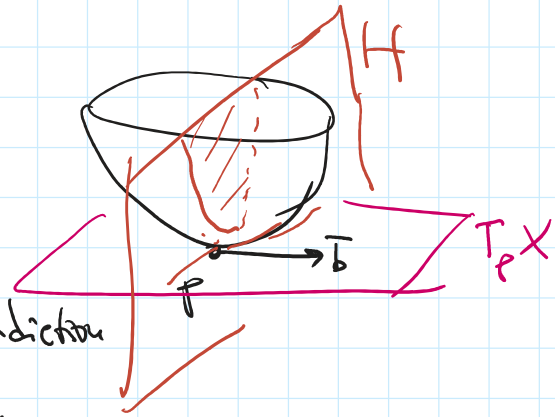
② Assume $n>0$, then by lemma $\dim T_pX > 0$

(Otherwise $T_pX = 0 \Rightarrow X = pt \Rightarrow \dim X = 0$)

Choose a vector $\bar{b} \neq 0$, $b \in T_p X$,

Also, choose a hyperplane $H \subset \mathbb{A}^n$ through p such that $b \notin H$.

Define $Y = X \cap H$



- $Y \neq \emptyset$ since $p \in Y$
- $Y \neq X$: indeed, if $X \subset H$ then $T_pX \subset H \Rightarrow b \in H$, contradiction

$\Rightarrow \dim$ (all components of Y) $= \dim X - 1$

Now $\dim T_pX \geq \dim T_pY + \underset{b}{1} \geq (\dim X - 1) + 1 = \dim X$

assumption ⊛ of induction

and we are done.

⊛ If $Y = \cup Y_i$ then $\dim Y_i = \dim X - 1$

$\dim T_pY \geq \dim T_pY_i \geq \dim Y_i = \dim X - 1$.

assumption of induct'n