Lecture 22 (3/1)

Thursday, February 29, 2024 7:49 PM

Recap: $\dim T_pX \geq \dim X$

Def Assume X irreducible. A point p is smooth if $\dim T_pX = \dim X$, and singular if $\dim T_pX > \dim X$.

$\mathrm{Sing}\, X = \{\text{singular points in } X\}$.

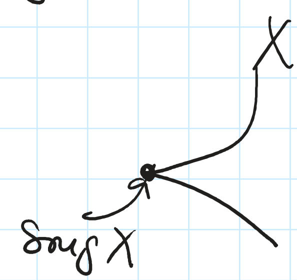
Note: We proved that $\{p : \dim T_pX \geq j\}$ is closed, so

- $\mathrm{Sing}\, X$ is closed in X.
- The set of smooth points is open in X.

Fact (w/o proof) Any X has at least one smooth point.

Cor The set of smooth points is non-empty, open $\Rightarrow$ dense in X!

Cor $\mathrm{Sing}\, X$ = proper closed subset of X, so $\dim \mathrm{Sing}\, X < \dim X$.



Example $X = \{f = 0\} \subset \mathbb{A}^n$, $f(x_1 \ldots x_n)$ = irreducible polynomial

We know $\dim X = n - 1$.

$$\boxed{\begin{aligned} &p \in X \text{ is smooth} \iff \dim T_pX = n-1 \iff \\ &\qquad \operatorname{rank}\left(\frac{\partial f}{\partial x_1} \; \cdots \; \frac{\partial f}{\partial x_n}\right) = 1 \\ &\iff \text{at least one of } \frac{\partial f}{\partial x_i}(p) \neq 0. \end{aligned}}$$

$p \in X$ is smooth $\iff \dim T_p X = n-1 \iff$
$$\operatorname{rank}\left(\frac{\partial f}{\partial x_1} \; \dots \; \frac{\partial f}{\partial x_n}\right) = 1$$
$\iff$ at least one of $\frac{\partial f}{\partial x_i}(p) \neq 0$.

$p \in X$ is singular if $\frac{\partial f}{\partial x_1}(p) = \dots = \frac{\partial f}{\partial x_n}(p) = 0$.

Let us prove X has at least one smooth point. [skip in class].

Assume not, then $\frac{\partial f}{\partial x_i}(p) = 0$ for all points $p \in X$ and all i

$\Rightarrow \frac{\partial f}{\partial x_i}$ is divisible by f.

Since $\deg_{x_i} \frac{\partial f}{\partial x_i} < \deg_{x_i} f$, we conclude $\frac{\partial f}{\partial x_i} = 0$.

In char 0, we get $\frac{\partial f}{\partial x_1} = \dots = \frac{\partial f}{\partial x_n} = 0 \Rightarrow f = \text{const}$.

In char p, we get that f is a polynomial in $x_1^p, \dots, x_n^p$
$$f(x_1 \dots x_n) = \sum a_{i_1 \dots i_n} x_1^{p i_1} \dots x_n^{p i_n}$$
But then $f = \left(\sum \sqrt[p]{a_{i_1 \dots i_n}} \, x_1^{i_1} \dots x_n^{i_n}\right)^p$, so

it is not irreducible, contradiction

* $\sqrt[p]{a_{i_1 \dots i_n}}$ exists since $\mathbb{K}$ alg. closed

Recall that a hyperplane in $\mathbb{P}^n$ is given by linear equation $H = \{a_0 x_0 + \dots + a_n x_n = 0\}$.

The set of all hyperplanes is the set of all $[a_0 : \dots : a_n]$ which is also a copy of $\mathbb{P}^n$, denoted by $(\mathbb{P}^n)^\vee$

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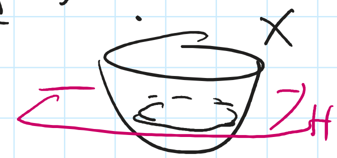

Bertini's Thm Suppose $X \subset \mathbb{P}^n$ irreducible, smooth. Then there exists an open dense subset $\mathcal{U} \subset (\mathbb{P}^n)^\vee$, such that for all $H \in \mathcal{U}$ the intersection $X \cap H$ is smooth.

Rmk In short, for H "in general position" the intersection $X \cap H$ is smooth, or H is transversal to X (see below). This is an analogue of Sard's theorem [$\mathcal{U}$ is the set of H "in general position"].

Proof: ① Pick $p \in X$, assume $H \ni p$. We have the following cases:

WLOG $p \in \{x_n \neq 0\} \subset \mathbb{P}^n$, so $x_n = 1$

- $X \subset H$, then $T_p X \subset T_p H = H$
- all components of $X \cap H$ have $\dim = d-1$ $\quad$ $\dim X = d$

p is smooth in $X \cap H \iff \dim T_p(X \cap H) = d-1$

If $f_1 \ldots f_k$ are equations of X, and $H = \{a_0 x_0 + \ldots + a_n = 0\}$

$X \cap H = \{f_1 = \ldots = f_k = a_0 x_0 + \ldots + a_n = 0\}$.

$$X \text{ smooth at } p \Rightarrow \operatorname{rank} \underbrace{\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_k}{\partial x_1} & \cdots & \frac{\partial f_k}{\partial x_1} \end{pmatrix}}_{J} = n-d$$

$X \cap H$ smooth at $\quad \begin{pmatrix} \frac{\partial f_1}{} & \frac{\partial f_1}{} \end{pmatrix}$ $\quad$ H is transversal to $T_p X$

$X \cap H$ smooth at p $\Leftrightarrow$ rank $\tilde{J} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \\ \frac{\partial f_k}{\partial x_1} & \cdots & \frac{\partial f_k}{\partial x_1} \\ a_0 & \cdots & a_{n-1} \end{pmatrix} = n-d+1.$ $\Leftrightarrow$ H is transversal to T_pX

$X \cap H$ singular at $p \Leftrightarrow$ rank $\tilde{J} = n-d =$ rank $J \Leftrightarrow$

$\Leftrightarrow (a_0, \ldots a_{n-1}) \in$ rowspan (J).

$\Leftrightarrow H \supset T_p X$

Conclusion: $X \cap H$ singular at p (or $X \subset H$) $\Leftrightarrow$ H is not transversal to T_pX

$\Leftrightarrow H \supset T_p X$.

(2) Consider the space of pairs

$$Z = \{(p, H) : p \in X, \ X \subset H \text{ or } X \cap H \text{ singular at } p\}$$

$$X \xleftarrow{\pi_1} Z \xrightarrow{\pi_2} (\mathbb{P}^n)^\vee$$

Given a point $p \in X$, the fiber $\pi_1^{-1}(p)$ is the set of hyperplanes H containing T_pX (*)

which is $\mathbb{P}\left(T_p\mathbb{P}^n / T_pX\right)^\vee \simeq \mathbb{P}^{n-d-1}$

(*) $L \subset \mathbb{A}^n$ linear subspace. dim $= d$. The set of hyperplanes containing L = the set of hyperplanes in $\mathbb{A}^n/L = \mathbb{A}^{n-d}$

$\Rightarrow \mathbb{P}^{n-d-1}$.

In fact, π_1 is a locally trivial fibration

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So $\dim Z = \dim X + \dim(\mathbb{P}^{n-d-1}) = d + n - d - 1 = n-1$

(3) Define $\mathcal{U} = (\mathbb{P}^n)^\vee \setminus \overline{\pi_2(Z)}$.

A hyperplane H is in the image $\pi_2(Z)$ iff there exists a point p such that $X \cap H$ is singular at p.

If H is <u>not</u> in $\pi_2(Z)$ then $X \cap H$ is smooth!

Now $\dim \overline{\pi_2(Z)} \leq \dim Z = n-1 < \dim(\mathbb{P}^n)^\vee$

So $\overline{\pi_2(Z)}$ is a proper closed subset of $(\mathbb{P}^n)^\vee$

$\Rightarrow \mathcal{U}$ is a nonempty dense open.

Rmk In fact, $\pi_2(Z)$ is closed but we do not know this yet.

Ex Suppose f is a degree d homogeneous polynomial in $x_0 \ldots x_n$.

If the coefficients of f are in general position then $\{f=0\}$ is smooth.

Proof Consider the Veronese embedding

$$\mathbb{P}^n \xhookrightarrow{\varphi} \mathbb{P}^N \qquad N = \#\text{all degree } d \text{ monomials in } x_0, \ldots x_n \; -1.$$

$\{f=0\}$ = intersection of the image of φ with a hyperplane H in $\mathbb{P}^N$.

If this hyperplane is generic $\varphi(\mathbb{P}^n) \cap H$ is smooth

If this hyperplane is generic, $\varphi(\mathbb{P}^n) \cap H$ is smooth by Bertini's thm.