Lecture 23 (3/4)

Sunday, March 3, 2024 8:34 PM

# Differentials

$A$ = algebra over $\mathbb{k}$

$\Omega_A$ = $A$-module generated by symbols $df$, $f \in A$

such that
- $d(f_1+f_2) = d(f_1) + d(f_2)$
- $d(c) = 0$ for $c \in \mathbb{k}$
- $d(fg) = f\,d(g) + g\,d(f)$

**Ex** $A = \mathbb{k}[x_1, \ldots x_n]$, by product rule one can check

$$df = \sum \frac{\partial f}{\partial x_i} dx_i$$

So $\Omega_A = A\langle dx_1, \ldots dx_n\rangle = \{\varphi_1 dx_1 + \cdots + \varphi_n dx_n\}$ $\varphi_i \in A$.

$\longleftrightarrow$ algebraic 1-forms on $\mathbb{A}^n$

**Ex** $X = \{f_1 = \cdots = f_k = 0\} \subset \mathbb{A}^n$

$$A(X) = \frac{\mathbb{K}[x_1 \ldots x_n]}{(f_1 \ldots f_k)}$$

Thus $\Omega_X = \Omega_{A(X)} = \dfrac{\mathbb{K}[x_1 \ldots x_n]\langle dx_1 \ldots dx_n\rangle}{(f_1 \ldots f_k, df_1, \ldots df_k)} = \dfrac{A(X)\langle dx_1 \ldots dx_n\rangle}{(df_1 \ldots df_k)}$

**Proof** • Again by product rule, $\Omega_X$ is generated over $A(X)$ by $dx_1, \ldots dx_n$. We need to describe the relations.

- Clearly, $f_i = 0$ in $A(X)$ and $d(0) = 0$, so $f_i = df_i = 0$ in $\Omega_{A(X)}$
- $\sum f_i g_i \in I(X) \Rightarrow$
$$d\left(\sum f_i g_i\right) = \sum f_i\, d(g_i) + \sum g_i\, df_i = 0$$
whenever $f_i = df_i = 0$. So these are all relations $\square$

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__Ex__ $\{x^2 = y^3\} = X$

$$\Omega_X = \frac{A(X)\langle dx, dy\rangle}{\langle 2x\,dx = 3y^2\,dy\rangle}$$

If $x \neq 0$ then $dx = \frac{3y^2}{2x}\,dy$

If $y \neq 0$ then $dy = \frac{2x}{3y^2}\,dx$

__Note:__ Outside $(0,0)$ $\Omega_X$ has rank 1 over $A(X)$

At $(0,0)$: rank 2

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__Universal property__ A derivation $\partial : A \to M$ valued in $A$-module $M$ is a $\mathbb{K}$-linear map such that

$$\partial(f_1 f_2) = f_1 \partial(f_2) + f_2 \partial(f_1)$$

__Fact__ Any derivation $\partial : A \to M$ corresponds to an $A$-linear map $\varphi : \Omega_A \to M$ such that

$$\begin{tikzcd} A \arrow[r, "d"] \arrow[dr, "\partial"'] & \Omega_A \arrow[d, "\varphi"] \\ & M \end{tikzcd} \qquad \varphi(f\,dg) = f\,\partial g$$

**Lemma** $m_p$ = maximal ideal at $p$

$$\Omega_X / m_p \Omega_X \simeq m_p / m_p^2 \underset{\text{from last week.}}{\simeq} (T_p X)^*$$

**Proof**: We define a map

$$m_p \longrightarrow \Omega_X / m_p \Omega_X$$

$$\varphi \longmapsto d\varphi$$

$$\varphi_1 \varphi_2 \longrightarrow d(\varphi_1 \varphi_2) = \varphi_1 d(\varphi_2) + \varphi_2 d(\varphi_1)$$

This vanishes mod $m_p \Omega_X$

if $\varphi_1, \varphi_2 \in m_p$.

Furthermore, if $f \in A(X)$ then $\overbrace{f - f(p)}^{=\varphi} \in m_p$

$$df = d(f - f(p)) + d(\underbrace{f(p)}_{\text{Constant}}) = d\varphi + 0 = d\varphi$$

If $g = g(p) + \varphi'$ then

$$g\,df = (g(p) + \varphi')\,d\varphi = g(p)\,d\varphi + \underbrace{\varphi'\,d\varphi}_{\in\, m_p \Omega_X}.$$

So $\Omega_X / m_p \Omega_X$ is spanned by $d\varphi$. $\boxtimes$

**Proof 2**: WLOG $p = (0, \dots, 0)$ $\quad A(X) = \dfrac{\mathbb{K}[x_1 \dots x_n]}{(f_1 \dots f_k)}$

$$m_p / m_p^2 = \frac{\operatorname{Span}(x_1 \dots x_n)}{(\text{linear parts of } f_i)} \qquad f_i = \underbrace{\sum a_{ij} x_j}_{\text{linear part.}} + \dots$$

$$df_i = \sum a_{ij}\,dx_j + \underbrace{\dots\dots}_{\in\, m_p \Omega_X}$$

$$df_i = \sum a_{ij}\, dx_j + \underbrace{\cdots\cdots}_{\in \mathfrak{m}_p \Omega_X}$$

$$\frac{\Omega_X}{\mathfrak{m}_p \Omega_X} = \frac{\mathrm{Span}(dx_1, \ldots, x_n)}{(\sum a_{ij}\, dx_j = 0)} \simeq \frac{\mathfrak{m}_p}{\mathfrak{m}_p^2}.$$

**Cor** $\Omega_X =$ algebraic 1-forms on $X$,

at every point we choose an element of the (Zariski) cotangent space $(T_p X)^*$.

**Rmk** Derivations $\partial : A \longrightarrow A$ (by universal property)

$\Updownarrow$

$A$-linear maps $\Omega_A \longrightarrow A$

At $p$: $\Omega_A / \mathfrak{m}_p \Omega_A \longrightarrow A/\mathfrak{m}_p \simeq \mathbb{K}$

$\Omega_A / \mathfrak{m}_p \Omega_A \simeq (T_p X)^*$

$\mathbb{K}$-Linear functionals on $(T_p X)^* = T_p X$

Choose a tangent vector at every point.

**Def** $X =$ projective variety, then

$$\Omega_X = \left\{ \begin{array}{l} \omega \text{ 1-forms in } dx_i \\ \text{with rational coefs} \end{array} : \begin{array}{l} \text{for every point } p \in X \\ \text{there is an open affine} \\ U \ni p \text{, such that} \\ \omega|_U \in \Omega_U \end{array} \right\}$$

**Ex** $X = \mathbb{P}^1$ with coord $[x_0 : x_1]$ $\quad \omega \in \Omega_{\mathbb{P}^1}$

<u>Ex</u> $X = \mathbb{P}^1$ with coord $[x_0 : x_1]$ $\quad \omega \in \Omega_{\mathbb{P}^1}$

Two charts: $\{x_0 \neq 0\}$ with $z = \frac{x_1}{x_0}$ $\rightsquigarrow$ $\omega = f(z)\,dz$

$\{x_1 \neq 0\}$ with $\frac{1}{z} = \frac{x_0}{x_1}$ $\rightsquigarrow$ $\omega = g\left(\frac{1}{z}\right) d\left(\frac{1}{z}\right)$.

So $f(z)\,dz = g\left(\frac{1}{z}\right)\left(-\frac{dz}{z^2}\right)$ $\qquad f, g = $ polynomials

$$z^2 f(z)\,dz = -g\left(\frac{1}{z}\right) dz \Rightarrow z^2 f(z) = -g\left(\frac{1}{z}\right)$$

This is not possible! LHS = polynomial in $z$ of degree $\geq 2$

So there are <u>no</u> global 1-forms on $\mathbb{P}^1$.