Lecture 24 (3/6)

Monday, March 4, 2024 12:57 PM

Curves and genus

Ex $X = \{x_1^2 x_2 = x_0(x_0 - x_2)(x_0 - 2x_2)\} \subset \mathbb{P}^2$ cubic curve.

In chart $x_2 \neq 0$, define $x = \frac{x_0}{x_2}$, $y = \frac{x_1}{x_2}$

$$X = \{y^2 = x(x-1)(x-2) = p(x)\}$$

Claim X is smooth

Proof At $\{x_2 \neq 0\}$, we get $y^2 - p(x) = 0$

$$J = \begin{pmatrix} -p'(x) & 2y \end{pmatrix}$$

If $y = 0$ then $p(x) = 0 \Rightarrow p'(x) \neq 0$ since $p(x)$ has distinct roots

so $\operatorname{rank}(J) = 1$ everywhere.

At $x_2 = 0$, we get $x_0^3 = 0 \Rightarrow$ one point at infinity $[0:1:0]$

In chart $\{x_1 \neq 0\}$, we get $x_2 = x_0(x_0 - x_2)(x_0 - 2x_2)$

Linearization at $(x_0 = x_2 = 0)$: $x_2 = 0 \Rightarrow \operatorname{rank}(J) = 1 \Rightarrow$ smooth at $[0:1:0]$

Now we want to study 1-forms on X

In the chart $\{x_2 \neq 0\}$ we get $y^2 = p(x)$, so

$$\boxed{2y\,dy = p'(x)\,dx}$$

Claim $\frac{dx}{y}$ defines a global regular 1-form on X.

$\{x_2 \neq 0\}$

- If $y \neq 0$, we are OK

- If $y = 0$ then $p(x) = 0$ ...

- If $y \neq 0$, we are OK
- If $y = 0$ then $p(x) = 0$, but $p'(x) \neq 0$ ($p(x)$ has simple roots)

Therefore we can rewrite $\frac{dx}{y} = \frac{2dy}{p'(x)}$

What happens at ∞? $x_1^2 x_2 = x_0(x_0 - x_2)(x_0 - 2x_2)$

$$\frac{dx}{y} = \frac{d\left(\frac{x_0}{x_2}\right)}{(x_1/x_2)} = \frac{x_2 dx_0 - x_0 dx_2}{x_2^2} : \left(\frac{x_1}{x_2}\right) =$$

$$= \boxed{\frac{x_2 dx_0 - x_0 dx_2}{x_1 x_2} = \omega}$$

In chart $x_1 = 1$: $x_2 = x_0(x_0 - x_2)(x_0 - 2x_2) = x_0^3 - 3x_0^2 x_2 + 2x_0 x_2^2$

$$dx_2 = (3x_0^2 - 6x_0x_2 + 2x_2^2)dx_0 + (-3x_0^2 + 4x_0x_2)dx_2$$

$$\boxed{dx_2 = \frac{(3x_0^2 - 6x_0x_2 + 2x_2^2)dx_0}{(1 + 3x_0^2 - 4x_0x_2)}}$$

$$\omega = \frac{x_2 dx_0 - x_0 dx_2}{x_2} = dx_0 - \frac{x_0 dx_2}{x_2} = dx_0 - \frac{(3x_0^3 - 6x_0^2 x_2 + 2x_0x_2^2)dx_2}{x_2}$$

but from the equation x_0^3 is divisible by x_2, so

$(x_0^3 = x_2 + 3x_0^2 x_2 - 2x_0 x_2^2)$

ω is regular whenever $1 + 3x_0^2 - 4x_0x_2 \neq 0$

This is an open subset containing the point at ∞ $[0:1:0]$.

So we are done!

Def X = smooth, projective curve ($\dim X = 1$)

The geometric genus $g(X)$ is defined as

Def X = smooth, projective curve (char $k = 0$)

The geometric genus $p_g(X)$ is defined as

$$p_g(X) = \dim \{\text{global 1-forms on } X\} = \Gamma(\Omega^1_X)$$

Ex $X = \mathbb{P}^1 \Rightarrow$ last time we proved there're no global 1-forms

$\Rightarrow p_g(\mathbb{P}^1) = 0$.

Ex $X = \{y^2 = x(x-1)(x-2)\} \subset \mathbb{P}^2$

Then $p_g(X) = 1$ and in fact

$$\Gamma(\Omega^1_X) = \mathrm{Span}\left(\frac{dx}{y}\right)$$

Here are some cool facts:

(1) $X_{\mathbb{C}}$ = smooth complex curve for $\mathbb{k} = \mathbb{C}$.

$\Rightarrow X_{\mathbb{C}}$ = smooth 2-dimensional surface in $\mathbb{CP}^2$.

$\mathbb{CP}^2$ is compact $\Rightarrow X_{\mathbb{C}}$ is a compact surface.

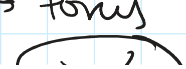
One can check it is orientable.

Classification of surfaces Any orientable, compact, connected surface is homeomorphic to a genus g surface

$g = 3$.

$$\boxed{\text{Thm } p_g(X_{\mathbb{C}}) = \text{genus of } X_{\mathbb{C}}}$$

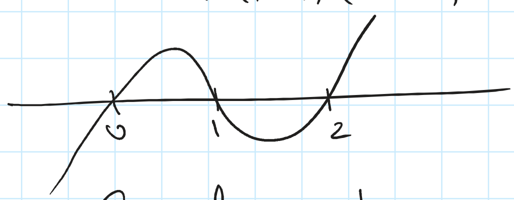
Ex $y^2 = x(x-1)(x-2) \longleftrightarrow g = 1 \longleftrightarrow$ torus in $\mathbb{CP}^2$.

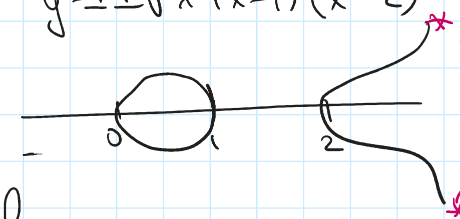


Ex $y^2 = x(x-1)(x-2) \longleftrightarrow g=1 \longleftrightarrow$ torus in $\mathbb{CP}^2$.

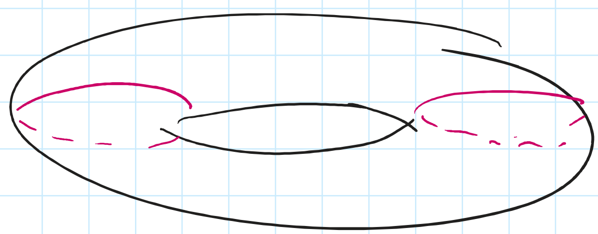
(2) $X_{\mathbb{R}}$ = smooth 1-dim curve in $\mathbb{RP}^2$

Ex $y^2 = x(x-1)(x-2)$





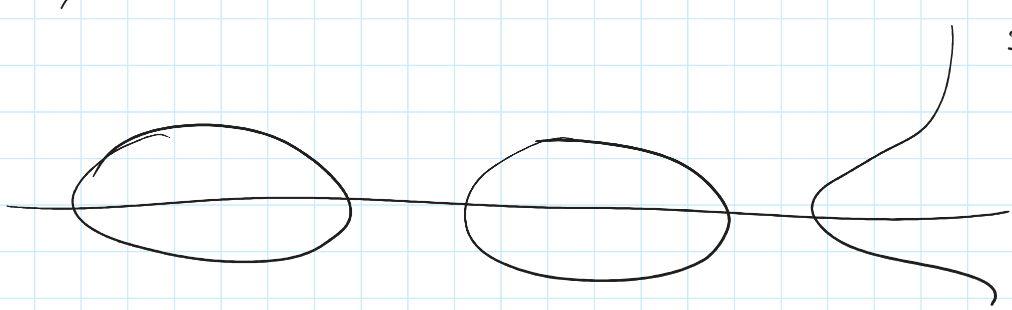
So the picture in $\mathbb{RP}^2$ looks like

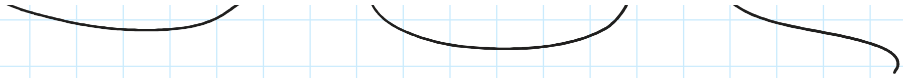


which is indeed a real cross-section of a complex curve $X_{\mathbb{C}}$.

Def Hyperelliptic curve in $\mathbb{A}^2$

$X = \{y^2 = p(x)\} \longleftarrow$ degree d polynomial with d distinct roots.





<u>Lemma</u> $\frac{dx}{y}$ is regular on this curve.

<u>Proof</u>: Same as above.

<u>Exercise</u>. If we consider the same equation homogenized in $\mathbb{P}^2$: $x_1^2 x_2^{d-2} = \tilde{p}(x_0, x_2)$ $\quad (*)$

then for $d > 3$ the point at ∞ $[0:1:0]$ is <u>singular</u>.

<u>Fact</u>: (1) There exists a smooth projective curve $\tilde{X}$ which resolves the singularity of $(*)$ at ∞.

(2) $p_g(X) = \begin{cases} k, & d = 2k+1 \\ k, & d = 2k+2. \end{cases}$ $\quad p_g(X) = g(X_{\mathbb{C}})$

$$\tilde{X} - \{\infty\} \cong X - \{\infty\}$$

In fact, the forms $\frac{dx}{y}, x\frac{dx}{y}, \ldots, x^{k-1}\frac{dx}{y}$

<u>extend</u> from $X - \{\infty\}$ to $\tilde{X}$.

and give a basis of $\Gamma(\Omega^1_{\tilde{X}})$.