Recap: X = smooth, projective curve ($\dim X = 1$)

$X(\mathbb{C})$ = smooth complex 1-dim manifold

= smooth, compact, orientable surface

$\Rightarrow$ genus g surface

Def The geometric genus of X:

$$p_g(X) = \dim\left\{\begin{matrix}\text{global regular}\\ \text{1-forms}\\ \text{on } X\end{matrix}\right\} = \Gamma(\Omega^1_X)$$

Fact $p_g(X) = \text{genus}(X(\mathbb{C}))$

Ex $X = \mathbb{P}^1$ $\quad X(\mathbb{C}) = \mathbb{C}P^1 \sim S^2 \quad g = 0$

Ex $X = \{y^2 = x(x-1)(x-2)\} \subset \mathbb{P}^2$

Last time: $p_g(X) = 1$, so $X(\mathbb{C})$ is a torus.

Why? How to see this topologically?

Recall: X = top. space (CW complex)

$$\chi(X) = \sum (-1)^i \#(i\text{-cells})$$ Euler characteristic

• Does not depend on cell decomposition

- Does not depend on cell decomposition
  $\Rightarrow$ invariant of X!
- If $Z \subset X$ closed then

$$\chi(X) = \chi(Z) + \chi(X \setminus Z) \quad \text{(additive)}$$

- If $Y \xrightarrow{\pi} X$ degree n covering map then $\chi(Y) = n\,\chi(X)$
- $\chi(\text{genus } g \text{ surface}) = 2 - 2g$

So we can determine the genus from χ!!

<u>Ex</u> $X \cap \mathbb{A}^2 = \{y^2 = x(x-1)(x-2)\}$

$$\downarrow \pi$$

$$\mathbb{A}^1_x$$

1) $\pi^{-1}(\{0\}), \pi^{-1}(\{1\}), \pi^{-1}(\{2\}) = 1$ point
2) $\pi^{-1}(x) = \pm\sqrt{x(x-1)(x-2)} = 2$ points
   for $x \neq 0, 1, 2$ (2 complex roots)

In fact, π is a degree 2 covering map over $\mathbb{C} \setminus \{0, 1, 2\}$ (prove it!)

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Now we can compute:

$$\chi(\mathbb{C}) = (-1)^2 = 1$$

$$\chi(\mathbb{C}\setminus\{0,1,2\}) = \chi(\mathbb{C}) - 3 = 1 - 3 = -2$$

$$\chi(\pi^{-1}(\mathbb{C}\setminus\{0,1,2\})) = 2\chi(\mathbb{C}\setminus\{0,1,2\}) = 2\cdot(-2) = -4$$

$$\chi(X \cap \mathbb{A}^2) = \chi(\pi^{-1}(\mathbb{C}\setminus\{0,1,2\})) + 3 = -4+3 = -1$$

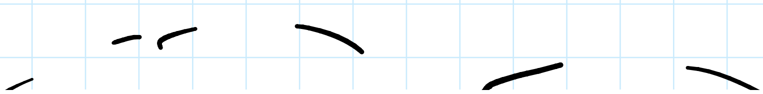
Finally, recall that X has one point $[0:1:0]$ at ∞, so $\chi(X) = -1+1 = 0$.

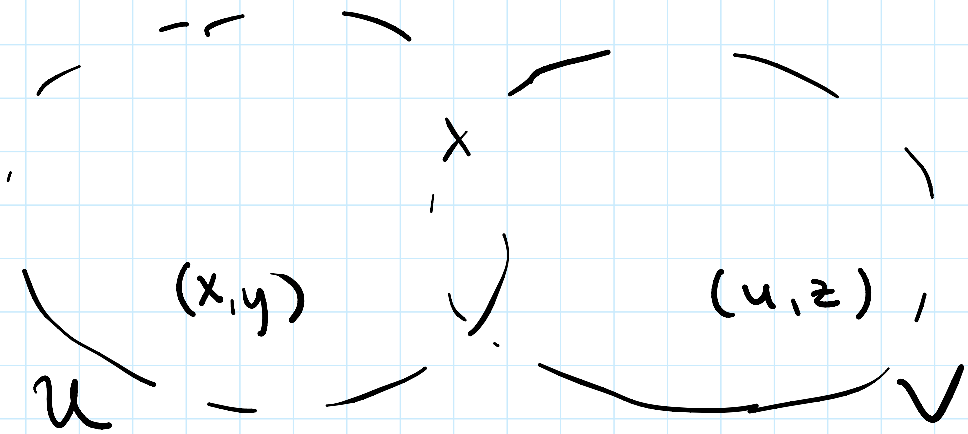
Now $0 = 2-2g$, so $g=1$.

<u>Hyperelliptic curves</u> Choose $p(x)$ = polynomial of degree d with distinct roots.

<u>Today</u>: assume d is <u>even</u>, odd is similar.

X is covered by two charts:





Transition functions: $u = 1/x$, $z = y/x^{d/2}$

Equations: $y^2 = p(x)$ in chart (x,y)

$$z^2 = u^d p(1/u)$$

Compatibility check: on $\mathcal{U} \cap \mathcal{V}$ we get

$$z^2 = \left(\frac{y}{x^{d/2}}\right)^2 = \frac{y^2}{x^d} = \frac{p(x)}{x^d} = u^d p(1/u) \quad \checkmark$$

Claim X is smooth

Proof: Check Jacobian in charts

(x,y): $J = \begin{pmatrix} -p'(x) & 2y \end{pmatrix}$ rk $= 1$

If $y=0$ then $p=0 \Rightarrow p'(x) \neq 0$ (simple root)

(u,z): same! Note $u^d p(1/u)$ also has distinct roots.

Gens from X: In chart (x,y) we get

$$\chi(X \cap \mathcal{U}) = 2\chi(\mathbb{C} \setminus \{\text{roots of } p\}) + \chi(\text{roots of } p)$$
$$= 2(1-d) + d = 2-d$$

How many points we missed?

Points "at ∞" are where $u=0$. Note $p(x) = x^d + \dots$

$\Rightarrow u^d p(1/u) = \dots + 1$. At $u=0$ we get $z^2 = 1$

so $z = \pm 1$.

$\boxed{\text{Two points at } \infty}$

Finally, $\chi(X) = 2 - d + 2 = 2 - 2g$

$$g = \frac{d-2}{2}$$

Genus from algebra:

$$y^2 = p(x) \qquad 2y\,dy = p'(x)\,dx$$

Define $\omega = \dfrac{dx}{y} = \dfrac{2dy}{p'(x)}$.

regular when $y \neq 0$

regular when $p'(x) \neq 0$

So regular everywhere in $\mathcal{U}$.

Change coords to the other chart:

$dx \quad d(1/u) \quad -du \quad u^{d/2} \quad z = y \quad d$

$$\omega = \frac{dx}{y} = \frac{d(1/u)}{(z/u^{d/2})} = -\frac{du}{u^2}\cdot\frac{u^{d/2}}{z}$$

$$\boxed{z = \frac{y}{x^{d/2}} = u^{\frac{d}{2}} y}$$

$$= -\frac{u^{\frac{d}{2}-2}\,du}{z}$$ ← regular when $z \neq 0$, covers pts at ∞ $(u,z) = (0, \pm 1)$

Furthermore,

$$x^k \omega = x^k \frac{dx}{y} = -\frac{u^{\frac{d}{2}-2}\,du}{z \cdot u^k}$$

always regular in (x, y) chart

regular in (u, z) chart when $k \leq \frac{d}{2} - 2$

$$y\omega = dx = -\frac{du}{u^2}$$ ← not regular!

So $T(\Omega^1_X) = \mathrm{Span}(\omega, x\omega, \dots, x^{\frac{d}{2}-2}\omega)$

$$\boxed{p_g(X) = \frac{d}{2} - 1 = \frac{d-2}{2} = g}$$

<u>Rmk</u> Here we did not use any specific embedding

Rmk There we did not use any specific embedding of X in $\mathbb{P}^N$. But this is OK.

Rmk We can consider the homogenized equation

$$x_1^2 x_2^{d-2} = \tilde{p}(x_0, x_2)$$

homogenization of $p(x) = p\left(\frac{x_0}{x_2}\right)$

$$\mathcal{U} = \left(x = \frac{x_0}{x_2}, \ y = \frac{x_1}{x_2}\right)$$

$$\mathcal{V} = \left(u = \frac{x_2}{x_0}, \ z = \frac{y}{x^{d/2}} = \frac{x_1 \, x_2^{\frac{d}{2}-1}}{x_0^{d/2}}\right)$$