Lecture 26 (3/11)

Sunday, March 10, 2024 7:32 PM

A bit more on hyperelliptic curves:

$X_{af} = X \cap \mathbb{A}^2$

$X_{af} = \{y^2 = p(x)\}$ has an automorphism

$\varphi(x, y) = (x, -y)$ $\boxed{\varphi^2 = Id}$

Regular, $(-y)^2 = y^2 = p(x)$ ✓ Defines $\mathbb{Z}_2$ action

Lemma $X_{af}/\mathbb{Z}_2 \simeq \mathbb{A}^1$

$X/\mathbb{Z}_2 \simeq \mathbb{P}^1$

Pf: $\pi : (x, y) \longrightarrow x$

$\pi^{-1}(x) =$ one $\mathbb{Z}_2$-orbit (clear id)

So we get a bijection $x \longleftrightarrow \mathbb{Z}_2$ orbit $(x, \pm\sqrt{p(x)})$. $\square$

Rmk Abstractly, assume $G =$ finite group acting on affine alg. set Y.

Then $A(Y/G) \simeq A(Y)^G$

and this can be used to define A/G as an algebraic variety.

In our case, $\mathbb{K}[x, y]^{\mathbb{Z}_2} = \mathbb{K}[x, y^2]$

In our case, $\mathbb{K}[x,y]^{\mathbb{Z}_2} = \mathbb{K}[x,y^2]$

$$A(X_{af}) = \frac{\mathbb{K}[x,y]}{(y^2 - p(x))} \qquad A(X_{af})^{\mathbb{Z}_2} = \frac{\mathbb{K}[x,y^2]}{(y^2-p(x))} = \mathbb{K}[x]$$

So $A(X_{af}/\mathbb{Z}_2) \simeq A(X_{af})^{\mathbb{Z}_2} \simeq A(\mathbb{A}^1)$

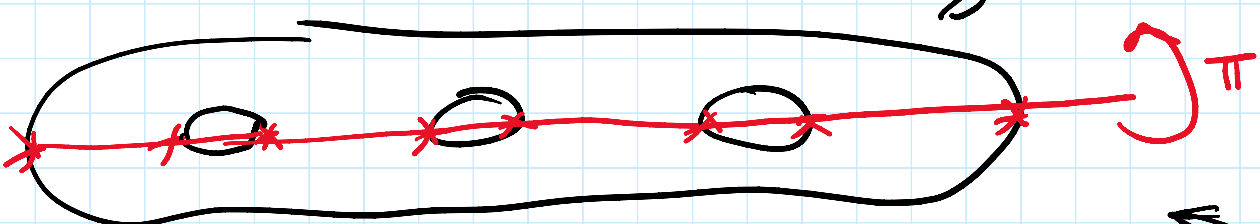
$\Rightarrow X_{af}/\mathbb{Z}_2 \simeq \mathbb{A}^1$. $\square$

Rmk The action of $\mathbb{Z}_2$ is not free!

fixed pts = (roots of $p(x)$, 0)

How to draw all this?

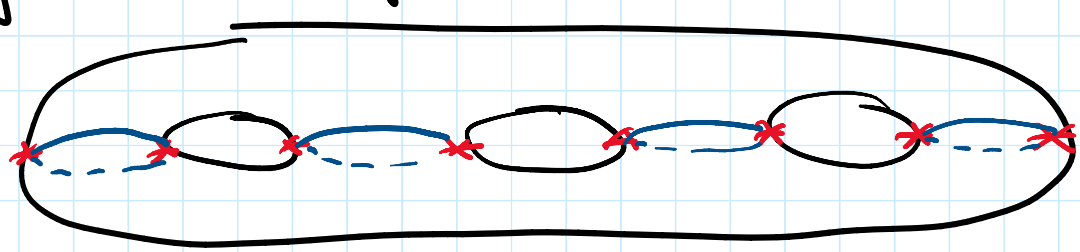
X = genus g surface



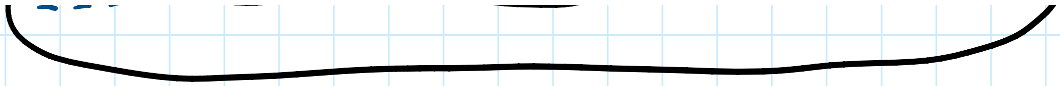
Put a genus g surface on a skewer, then rotation by π is an automorphism with $2g+2$ fixed points.

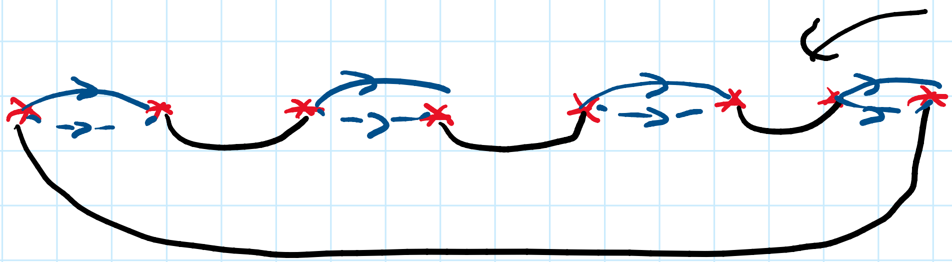
$X/\mathbb{Z}_2$:

Cut in half

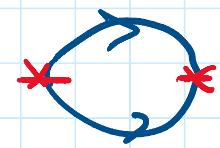


Cut in half

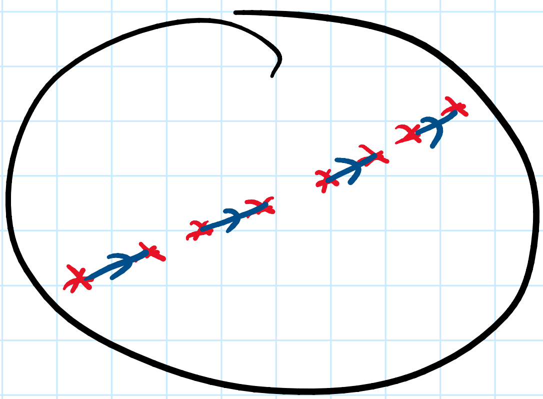




For each hole, we need to stitch two sides



and get a sphere with g+1 "cuts"



So $X/\mathbb{Z}_2 \simeq \mathbb{CP}^1$

as expected.

Divisors and Picard group.

Pic X → for X affine, characterizes whether $A(X)$ is a UFD

Pic X → for any X controls line bundles on X.

Def A (Weil) divisor on X is a linear combination $\sum a_i [D_i]$ where

D_i closed irreducible subsets in X

D_i = closed, irreducible subsets in X

and $\dim D_i = \dim X - 1$.

Ex $X = \mathbb{A}^n$, $f \in \mathbb{K}[x_1 \ldots x_n]$

Since $\mathbb{K}[x_1 \ldots x_n]$ is UFD, we can write

$f = f_1^{m_1} \cdots f_s^{m_s}$ $\qquad f_i$ = irreducible distinct

Define $\boxed{\operatorname{div}(f) = \sum m_i [\{f_i = 0\}]}$

Note $\{f_i = 0\}$ irreducible, closed, $\dim = \dim X - 1$.

We want to generalize this example.

Thm X = irreducible affine alg. set

Then the following are equivalent:

(a) $A(X)$ is a UFD

(b) All closed, irreducible subsets of X of $\dim = \dim X - 1$ are hypersurfaces $\{f = 0\}$.

Proof: (a) $\Rightarrow$ (b) Assume $A(X)$ is a UFD

$Y \subset X$ irred, $\dim Y = \dim X - 1$.

$I(Y)$ = prime ideal in $A = A(X)$

$I(Y) = \underline{\text{prime}}$ ideal in $A = A(X)$

Pick $f \in I(Y)$, since A is a UFD

we can factor $f = \prod f_i^{m_i}$, f_i = irred.

$X \supset \{f_i = 0\} \supset Y$ | $f \in I(Y) \Rightarrow f_i \in I(Y)$ for all i

$\dim \{f_i = 0\} = \dim X - 1$

If $Y \neq \{f_i = 0\}$ then $\dim Y < \dim \{f_i = 0\} < \dim X - 1$ contradiction.

$\Rightarrow Y = \{f_i = 0\}$.

(b) $\Rightarrow$ (a) Recall that f is called irreducible

if $xy = f \Rightarrow x$ or y is a unit. We need to prove that any f is a $\underline{\text{unique}}$ product of irreducibles.

- Assume f is not irreducible, then $f = xy$ $\leftarrow$ not units, and we can continue. By $\underline{\text{Noetherian}}$ property this process stops, and f = product of irreducibles.
- Assume f is irreducible, let us prove (f) is prime. Indeed, define $Y = \{f = 0\} = \bigcup Y_i$ = irred. comp.

We have $\dim Y_i = \dim X - 1$, so

[illegible]

We have $\dim Y_i = \dim X - 1$, so

by assumption (b) $I(Y_i) = (g_i)$.

$f \in I(Y_i) \Rightarrow f \in (g_i) \Rightarrow f = g_i \cdot k_i$

Since f is irreducible and g_i are not units, k_i is unit $\Rightarrow (f) = (g_i)$ is prime.

- Now it is easy: assume

$$f_1^{m_1} \cdots f_s^{m_s} = \bar{f}_1^{n_1} \cdots \bar{f}_t^{n_t} \quad \text{where } f, \bar{f} \text{ irred.}$$

Since (f_1) is prime, and RHS is in (f_1), we get one of $\bar{f}_i \in (f_1) \Rightarrow \bar{f}_i = f_1 \cdot \text{unit}$.

We can cancel and proceed by induction.

This shows uniqueness. $\square$