Lecture 27 (3/12)

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Suppose X is smooth* and irreducible

* In fact, it is sufficient for it to be normal

Fact Suppose X is smooth & affine

- $Y \subset X$ irreducible, $\dim Y = \dim X - 1$
- $f \in A(X)$, $f \neq 0$

Then one can define the order of vanishing (also known as divisorial valuation) $\nu_Y(f)$ with the following properties:

① $\nu_Y(f) \geq 0$

② If f identically vanishes on Y, that is $f \in I(Y)$ then $\nu_Y(f) > 0$. Otherwise $\nu_Y(f) = 0$.

③ $\nu_Y(fg) = \nu_Y(f) + \nu_Y(g)$

④ $\nu_Y(f+g) \geq \min(\nu_Y(f), \nu_Y(g))$.

⑤ If $(f) = I(Y)$ then $\nu_Y(f) = 1$

⑥ If $\nu_Y(f) \geq \nu_Y(g)$ then $\frac{f}{g}$ = regular function on open subset of Y

That is, $\frac{f}{g} = \frac{f'}{g'}$, $g' \notin I(Y)$.

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(w/o proof).

Ex $Y = \{y=0\} \subset \mathbb{A}^2_{x,y}$

$f = a_0(x) + y\,a_1(x) + y^2 a_2(x), \ldots$

$\nu_Y(f) = \min\{i : a_i(x) \neq 0\}$.

$\nu_Y(x+y) = 0 \qquad \nu_Y(xy^2) = 2$

Ex $f \in \mathbb{K}[x_1 \ldots x_n]$, $Y \subset \mathbb{A}^n = X$
irred, $\dim Y = n-1$

$f = \prod f_i^{m_i}$ $\quad f_i$ = irreducible

$$\nu_Y(f) = \begin{cases} m_i, & Y = \{f_i = 0\} \\ 0, & \text{otherwise}. \end{cases}$$

Def A Weil divisor on X

is $\sum a_i [Y_i]$ $\quad Y_i$ = irreducible, $\dim Y_i = \dim X - 1$

$a_i \in \mathbb{Z}$.

Def $f \neq 0$

$f \in A(X)$ $\quad \operatorname{div}(f) = \sum_Y \nu_Y(f)\,[Y]$

Note Only finitely many Y have $\nu_Y(f) \neq 0$

indeed Y should be a component of $\{f=0\}$.

Def $\frac{f}{g}$ = rational function on X $f, g \neq 0$

$$\mathrm{div}\left(\frac{f}{g}\right) = \mathrm{div}(f) - \mathrm{div}(g).$$

Lemma (a) For all f, g $\mathrm{div}(fg) = \mathrm{div}(f) + \mathrm{div}(g)$

(b) $\mathrm{div}\left(\frac{f}{g}\right)$ is well defined

homomorphism $\left[\underset{\text{rat. funct}}{\mathrm{Frac}\,A(X)^*}, \cdot\right] \longrightarrow \left(\mathrm{Div}(X), +\right)$

Proof (a) $\mathrm{div}(fg) = \sum_Y \nu_Y(fg)[Y] = \sum_Y (\nu_Y(f) + \nu_Y(g))[Y]$

$$= \sum_Y \nu_Y(f)[Y] + \sum_Y \nu_Y(g)[Y] = \mathrm{div}(f) + \mathrm{div}(g)$$

(b) $\frac{f}{g} \sim \frac{fh}{gh}$ $\mathrm{div}(fh) - \mathrm{div}(gh) =$

$$= \mathrm{div}(f) + \mathrm{div}(h) - \mathrm{div}(g) - \mathrm{div}(h)$$

$$= \mathrm{div}(f) - \mathrm{div}(g) = \mathrm{div}\left(\frac{f}{g}\right).$$

Similarly $\mathrm{div}\left(\frac{f}{g} \cdot \frac{f'}{g'}\right) = \mathrm{div}\left(\frac{f}{g}\right) + \mathrm{div}\left(\frac{f'}{g'}\right)$.

$\square$

Def A principal divisor is a divisor of the form $\mathrm{div}(f/g)$ f/g = rational fn on X.

By lemma, these form a subgroup $\mathrm{Prin}(X) \subset \mathrm{Div}(X)$.

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Def Divisor class group

$$Cl(X) = \frac{\mathrm{Div}(X)}{\mathrm{Prin}(X)}$$ ← all divisors ← principal divisors.

Ex $\mathrm{Div}(\mathbb{A}^1) = \{\sum a_i [p_i],\ p_i = \text{some points},\ a_i \in \mathbb{Z}\}$.

We can write $\sum a_i [p_i] = \sum a_i^+ [p_i] - \sum a_i^- [p_i]$ (positive, negative.)

$$\frac{f}{g} = \frac{\prod (x-p_i)^{a_i^+}}{\prod (x-p_i)^{a_i^-}} \quad \text{rational fn.}$$

$\mathrm{Div}\left(\frac{f}{g}\right) = \sum a_i [p_i]$ so $\mathrm{Div}(\mathbb{A}^1) = \mathrm{Prin}(\mathbb{A}^1)$

$Cl(\mathbb{A}^1) = 0$.

Ex What about $\mathbb{P}^1$? $\mathrm{Div}(\mathbb{P}^1) = \{\sum a_i [p_i]\}$ similar.

$\frac{f}{g}$ = rational function on $\mathbb{P}^1$ $\Leftarrow\Rightarrow$ $\frac{f(x_0, x_1)}{g(x_0, x_1)}$ ← same degree.

$\mathrm{Div}\left(\frac{f}{g}\right) = \sum(\text{roots of } f) - \sum(\text{roots of } g)$

- $d = \deg(f)$ = sum of multiplicity of roots of f

  = sum of multiplicity of roots of g

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So if $\mathrm{Div}\left(\frac{f}{g}\right) = \sum a_i [p_i]$ then $\sum a_i = 0$.

And conversely, if $\sum a_i = 0$, we can find a rational fn.

(prove it!)

<u>Claim</u> $\boxed{Cl(\mathbb{P}^1) = \mathbb{Z}}$

<u>Proof</u> $\mathrm{Div}(\mathbb{P}^1) \xrightarrow{\deg} \mathbb{Z}$ "degree"

$$\sum a_i [p_i] \xrightarrow{\deg} \sum a_i$$

$\mathrm{Ker}(\deg) = \mathrm{Prin}(\mathbb{P}^1)$, by Isomorphism Thm

$$Cl(\mathbb{P}^1) = \frac{\mathrm{Div}(\mathbb{P}^1)}{\mathrm{Prin}(\mathbb{P}^1)} = \mathbb{Z}$$

$\square$

<u>Thm</u> X = affine, smooth. *

$A(X)$ is a UFD $\Longleftrightarrow$ $Cl(X) = 0$

<u>Cor</u> $Cl(\mathbb{A}^n) = 0$.

<u>Proof</u>: Last time we proved (*)

$A(X)$ is a UFD $\Leftrightarrow$ any $Y \subset X$ closed, irred, $\dim Y = \dim X - 1$ satisfies $I(Y) = (f)$

① Assume $A(X) = \mathrm{UFD}$

$D = \sum a_i [Y_i]$

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By $(*)$ $I(Y_i) = (f_i)$, $\mathrm{div}(f_i) = [Y_i]$

then $D = \mathrm{div}(\prod f_i^{a_i})$ principal $\Rightarrow$ $Cl = 0$.

(2) Assume $Cl(X) = 0$, so any divisor is principal.

Pick $Y \subset X$ closed, irred, $\dim Y = \dim X - 1$

$[Y] = \mathrm{div}\left(\frac{f}{g}\right)$ for some f, g.

Now: $\nu_Y(f) - \nu_Y(g) = 0 \Rightarrow$ by (6) $\frac{f}{g} = \frac{f'}{g'}$ $g' \notin I(Y)$

$U_0 = \{g' \neq 0\} \cap Y$, $\underbrace{Z_0 = Y \cap \{g' = 0\}}$ $\dim Z_0 = \dim Y - 1$
$= \dim X - 2$

$\{g = 0\} = \bigcup Y_i$ $\leftarrow$ irred. components

for all i, $\nu_{Y_i}(f) = \nu_{Y_i}(g) \Rightarrow \frac{f}{g} = \frac{f^{(i)}}{g^{(i)}}$ $g^{(i)} \notin I(Y_i)$
$Y_i \neq Y$

Similarly, $Y_i \cap \{g^{(i)} = 0\}$ have $\dim = \dim X - 2$

So $\frac{f}{g}$ is regular outside of some subset of codim 2.

Since X is smooth, $\frac{f}{g}$ is regular everywhere.

So $Y = \{\varphi = 0\}$ for a regular fn $\varphi = \frac{f}{g}$. $\square$

Thm $Cl(\mathbb{P}^n) = \mathbb{Z}$.

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Proof $Y \subset \mathbb{P}^n$ irred, $\dim Y = n-1$

$\rightsquigarrow \tilde{Y} \subset \mathbb{A}^{n+1}$ irred, $\dim \tilde{Y} = n$

Since $\mathbb{A}^{n+1}$ is UFD, $\tilde{Y} = \{g = 0\}$ for some homogeneous irred. polynomial g.

same for Y.

Define $\deg(Y) = \deg(g)$.

$$\deg : \mathrm{Div}(\mathbb{P}^n) \longrightarrow \mathbb{Z}$$

$$\sum a_i [Y_i] \longrightarrow \sum a_i \deg(Y_i)$$

$\mathrm{Prin}(\mathbb{P}^n) \leftrightarrow$ rat. functions of degree 0 $= \mathrm{Ker}(\deg)$

$\prod g_i^{a_i}$ where g_i = equation of Y_i

Again by isomorphism thm, $\underbrace{\dfrac{\mathrm{Div}(\mathbb{P}^n)}{\mathrm{Prin}(\mathbb{P}^n)}}_{Cl(\mathbb{P}^n)} \simeq \mathbb{Z}$.