$$X = \{ y^2 z = x(x-z)(x-2z) \} \subset \mathbb{P}^2$$

__Recall__ X = smooth curve, one point at infinity $[0:1:0] = P_0$

$\mathrm{Div}(X) = \{ \sum a_i p_i \}$ divisors

$\mathrm{Prin}(X) = \{ \mathrm{div}\left(\frac{f}{g}\right) : f, g = \text{polynomials in } x, y, z \text{ of same degree} \}$

$$\mathrm{Cl}(X) = \frac{\mathrm{Div}(X)}{\mathrm{Prin}(X)} = \text{divisor class group.}$$

__Goal for today__: describe $\mathrm{Cl}(X)$.

(1) __Lemma__ We have a well-defined map

$$\deg : \mathrm{Cl}(X) \longrightarrow \mathbb{Z}$$
$$\sum a_i p_i \longrightarrow \sum a_i$$

__Proof__ We need to prove that for any rational function $\varphi = \frac{f}{g}$ $\deg(\mathrm{div}(\frac{f}{g})) = 0$.

So $\mathrm{Div} \xrightarrow{\deg} \mathbb{Z}$, $\mathrm{Prin} \subset \mathrm{Div}$, $\mathrm{Prin} \longrightarrow 0$ $\Rightarrow$ $\frac{\mathrm{Div}}{\mathrm{Prin}} \xrightarrow{\deg} \mathbb{Z}$ well defined.

Prin $\longrightarrow 0$ Prin well defined.

There are many proofs, here is a fancy one:

Bezout Thm Suppose F, G homog. polynomials in x, y, z of deg $= m, n$

Then $\{F=0\} \cap \{G=0\} = mn$ points with multiplicities.

Assume now $\deg f = \deg g = d$

Then $\deg(\mathrm{div}(f)) = \# X \cap \{f=0\}$

$= 3d$

$\deg(\mathrm{div}(g)) = 3d$

$\Rightarrow \deg\left(\mathrm{div}\left(\frac{f}{g}\right)\right) = 3d - 3d = 0.$

Note: Hartshorne has different pf. II.6.9 – II.6.10 $\square$

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(2) Define $Cl^0 = \mathrm{Ker}(\deg : Cl \longrightarrow \mathbb{Z})$

classes of divisors of degree 0

By Isomorphism Thm: $Cl / Cl^0 \simeq \mathbb{Z}$

or $0 \to Cl^0 \to Cl \to \mathbb{Z} \to 0$

So Cl is an extension between abelian groups Cl^0 and $\mathbb{Z}$.

(3) We are going to define map

Φ ...

We are going to define maps

$$X \xrightarrow{\Phi} Cl^\circ X$$
$$p \xrightarrow{\Phi} [p]-[p_0]$$

Recall $p_0=[0:1:0]$ point at ∞.

<u>**Thm**</u> ① Φ is a bijection (!)

② X has a group structure (since $Cl^\circ X$ is a group!)

such that $Cl^\circ X \simeq X$.

In particular, $Cl^\circ X$ is $\boxed{\text{large}}$

$\mathbb{K}=\overline{\mathbb{K}}$

char $\neq 2,3$

③ Suppose $\mathbb{K}$ is not alg. closed, char $\neq 2,3$ (ex: $\mathbb{R}$ or $\mathbb{Q}$). Then $\mathbb{K}$-points of X form a subgroup of $\overline{\mathbb{K}}$-points of X.

<u>**Ex**</u> Rational points on a cubic curve in $\mathbb{P}^2$ form a group called Mordell-Weil group of X.

<u>**lemma**</u> Φ is injective.

<u>**Proof**</u>: Assume $[p]-[p_0]=[q]-[p_0]$ in $Cl^\circ(X)$

(sketch) Then $[p]-[q]=\operatorname{div}\left(\frac{f}{g}\right)$ for some rational fn.

rational fn.

Define a map $X \xrightarrow[\varphi]{[f:g]} \mathbb{P}^1$

well defined since $\deg f = \deg g$.

<u>At p</u>: $\nu_p f - \nu_p g = 1$

$\Rightarrow \frac{f}{g} = \frac{f'}{g'}$ where $g'(p) \neq 0$, $\nu_p g = 0$

and $\nu_p f' = 1$

<u>At q</u>: $\nu_q f - \nu_q g = -1$, so symmetrically.

$\frac{f}{g} = \frac{f'}{g'}$ where $f'(q) \neq 0$, $\nu_q f = 0$

$\nu_q(g) = 1$.

At <u>all other points</u> z: $\nu_z f - \nu_z g = 0$

$\Rightarrow \frac{f}{g} = \frac{f'}{g'}$ where $f'(z) \neq 0$, $g'(z) \neq 0$ $\Rightarrow \varphi(z) \neq 0, \infty$.

Therefore $\varphi^{-1}(0) = p$ (with multiplicity 1)

$\varphi^{-1}(\infty) = q$ (with multiplicity 1).

This implies (Hartshorne. II.6.10.1)

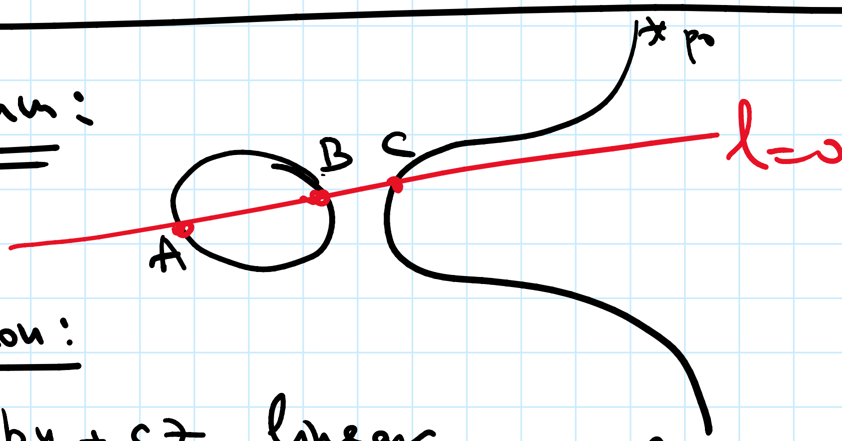
that φ is an isomorphism $\Rightarrow X \simeq \mathbb{P}^1$.

But we know that $p_g(X) = 1 \neq p_g(\mathbb{P}^1) = 0$,

so $X \neq \mathbb{P}^1$, contradiction. ∎

So $A \neq H$, contradiction.

Proof of Thm:



Construction:

$\ell = ax + by + cz$ linear function $\leftarrow$ line in $\mathbb{P}^2$

$\frac{\ell}{z}$ = well defined rational function.

What's $\mathrm{div}\left(\frac{\ell}{z}\right) = ?$

- $\{\ell = 0\} \cap X = 3$ points, call A, B, C
- $\{z = 0\} \cap X = \{p_0\}$ with multiplicity 3

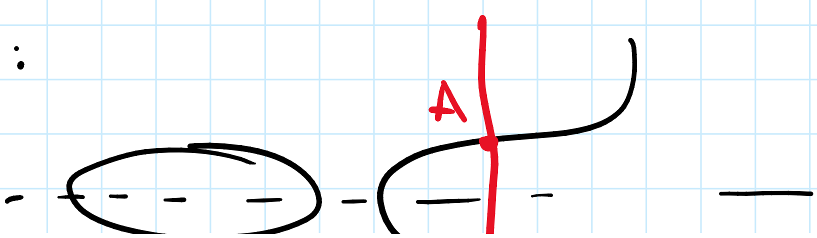
So $\mathrm{div}\left(\frac{\ell}{z}\right) = A + B + C - 3p_0$

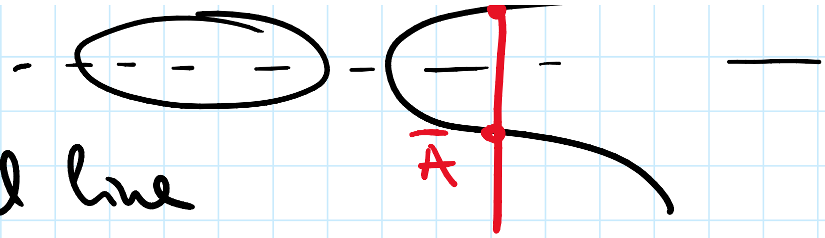
$$= (A - p_0) + (B - p_0) + (C - p_0).$$

Conclusion: If A, B, C on the same line

then $\underbrace{(A - p_0)}_{\Phi(A)} + \underbrace{(B - p_0)}_{\Phi(B)} + \underbrace{(C - p_0)}_{\Phi(C)} = 0$ in $\mathrm{Cl}^0(X)$.

Special case:





For a vertical line

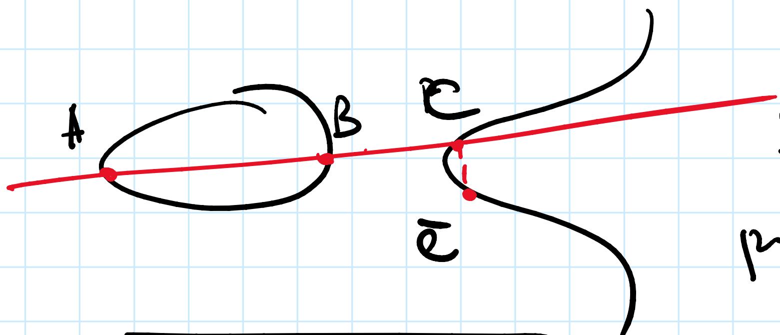
$$\{l=0\} \cap X = A + \bar{A} + p_0$$

$$\operatorname{div}\left(\frac{l}{z}\right) = A + \bar{A} + p_0 - 3p_0 = (A - p_0) + (\bar{A} - p_0).$$

So if $A, \bar{A}$ are symmetric around x-axis then

$$\Phi(A) + \Phi(\bar{A}) = 0.$$

Reverse the logic: given A, B on X,



consider line $\overline{AB} = \ell$

It intersects X in 3 points A, B, C, and we define

$$\boxed{A + B = \bar{C}}$$

where $\bar{C}$ = reflection of C.

This is the group law on X.

- $p_0 = O$ identity element since $\Phi(p_0) = 0$
- $\bar{C}$ = inverse of C since $\Phi(C) + \Phi(\bar{C}) = 0$.
- Associativity:

$$\Phi((A+B)+C) = \Phi(A+B) + \Phi(C) = \Phi(A) + \Phi(B) + \Phi(C)$$

$$\parallel \ \Phi(A + (B + C))$$

$$\Phi((A+B)+C) = \Phi(A+B) + \Phi(C) = \Phi(A)+\Phi(B)+\Phi(C)$$

$$\parallel \quad \Phi(A+(B+C))$$

Since Cl^0 is associative and Φ is injective, group law on X is associative.

Let us prove that Φ is surjective.

Suppose $\sum a_i p_i \in \mathrm{Div}^0(X)$ $\quad \sum a_i = 0$.

Then $\sum a_i p_i = \sum a_i (p_i - p_0) = \sum a_i \Phi(p_i)$.

We can assume all $a_i \geq 0$ (otherwise change $\underset{a_i \to -a_i}{}$ p_i to $\overline{p_i}$).

But then (inductively) $\sum a_i \Phi(p_i) = \Phi(\sum a_i p_i)$ and we are done.

Finally, what if $\mathbb{K}$ is not alg. closed?

This is fine: if A, B are defined over $\mathbb{K}$ (think $\mathbb{Q}$) then ℓ is defined over $\mathbb{K}$,

$X \cap \ell$ = cubic equation with 2 roots in $\mathbb{K}$

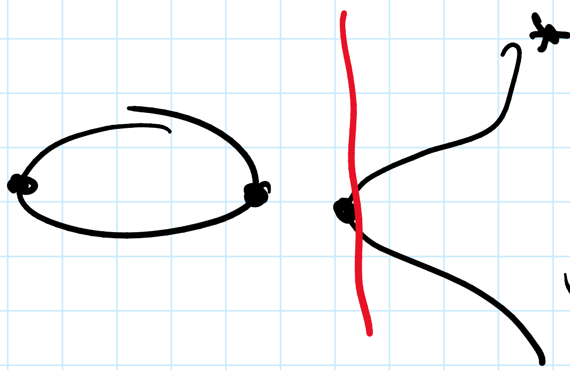
sum of the roots in $\mathbb{K}$ $\Rightarrow$ 3rd root in $\mathbb{K}$

$\Rightarrow$ C and $\overline{C}$ are defined over $\mathbb{K}$.

$\square$

Bonus: Which points on X have order 2?

$A + A = 0 \iff A = \bar{A}$



* these are precisely the branch points, or the roots of $p(x) = x(x-1)(x-2)$.

Further directions:

① **Thm** For any smooth X,

$$Cl(X) \simeq (\mathrm{Pic}\, X, \otimes)$$

iso classes of line bundles on X

$\mathcal{L} \otimes \mathcal{M}$ is a line bundle if $\mathcal{L}, \mathcal{M}$ are.

$Cl\, X / \mathrm{Pic}\, X$ breaks into "discrete" and "continuous" part.

$$\deg : Cl(X) \longrightarrow H^2(X;\mathbb{Z})$$
$$\sum a_i [Y_i] \longrightarrow \sum a_i \, PD[Y_i]$$

Also
$$\mathrm{Pic}(X) \longrightarrow H^2(X;\mathbb{Z})$$
$$\mathcal{L} \longrightarrow c_1(\mathcal{L})$$ first Chern class.

$\mathcal{L} \longrightarrow c_1(\mathcal{L})$ first Chern class.

Can define $Cl^0 \simeq Pic^0 = \ker(\deg)$; these are interesting.

2) **Thm** For X = smooth genus g curve

$Cl^0 X$ = projective algebraic variety

$\quad \|$

$Pic^0 X$ $\quad$ dim $= g$, has a structure of abelian group.

$\quad \|$

Jacobian of X $\quad$ (abelian variety).

Over $\mathbb{C}$, $Cl^0(X) \simeq 2g$-dimensional torus T^{2g}

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