Lecture 3 (1/12)

Wednesday, January 10, 2024 1:00 PM

Hilbert's Nullstellensatz

Assume $\mathbb{K}$ algebraically closed.

Thm 1 All maximal ideals in $\mathbb{K}[x_1 \ldots x_n]$ are of the form $(x_1 - a_1, \ldots, x_n - a_n)$

Thm 2 $I \subset \mathbb{K}[x_1 \ldots x_n]$ proper ideal $\Rightarrow Z(I) \neq \emptyset$

Thm 3 $I \subset \mathbb{K}[x_1 \ldots x_n]$ some ideal $f(x_1 \ldots x_n)$ vanishes on $Z(I)$. Then $f \in \sqrt{I}$.

Proof of Thm 3 $J \subset \mathbb{K}[x_1 \ldots x_n, x_{n+1}]$

$$J = I[x_{n+1}] + (x_{n+1} f - 1)$$

Last time: $Z(J) = \emptyset \Rightarrow J = \mathbb{K}[x_1 \ldots x_{n+1}]$
$\Rightarrow 1 \in J$

$$1 = g(x_1 \ldots x_n, x_{n+1}) + h(x_1 \ldots x_n, x_{n+1})(x_{n+1} f - 1)$$

g = poly in x_{n+1} w. coefs in I $= \sum g_i x_{n+1}^i$

$h \in \mathbb{K}[x_1 \ldots x_{n+1}]$

Plug in $x_{n+1} = \frac{1}{f}$, get identity of rational functions:

$$1 = \sum g_i \left(\frac{1}{f}\right)^i + \cancel{h(x_1 \ldots x_n, \tfrac{1}{f}) \cdot 0}$$

Clear denominators: $f^N = \sum g_i f^{N-i} \in I$.

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We will prove Theorem 1 in the rest of the lecture, but we need some commutative algebra first.

Def $K \subset L$ rings. We say that $z \in L$ is algebraic over K if

$$z^n + a_1 z^{n-1} + \dots + a_n = 0$$

for some $a_i \in K$.

Fact Algebraic elements of L form a ring

Lemma K = field, $z \in K(x)$ and

$$z^n + a_1 z^{n-1} + \dots + a_n = 0 \quad \text{for } a_i \in K[x].$$

Then $z \in K[x]$.

Proof $z = \frac{f}{g}$, $GCD(f,g) = 1$.

$$\left(\frac{f}{g}\right)^n + a_1 \left(\frac{f}{g}\right)^{n-1} + \dots + a_n = 0$$

$$f^n + \underbrace{a_1 f^{n-1} g + \dots + a_n g^n}_{} = 0$$

$$\underbrace{f^n + a_1 f^{n-1} g + \dots + a_n g^n}_{\text{divisible by } g} = 0 \Rightarrow \underline{g = 1}.$$

coprime to g.

$\square$

Thm $K \subset L$ fields, and L is finitely generated over K

That is, there exist $v_1 \dots v_n \in L$ such that any element of L can be written as a polynomial in $v_1 \dots v_n$ with coefs in K. Then any elt of L is algebraic over K.

Proof Induction in n.

$(n=1)$ $K[x] \twoheadrightarrow L$ surjective. $x \mapsto v_1$

$K[x]$ is not a field $\Rightarrow L = \frac{K[x]]}{I}$ $\quad I = (f)$
PID $\quad f \neq 0$

So $L = \frac{K[x]}{(f)}$ and L is algebraic.

$(n>1)$ L is generated by $v_2 \dots v_n$ over $K(v_1)$ field

$\Rightarrow$ by the assumption & induction L is algebraic over $K(v_1)$ $\leftarrow$ field

Case 1 v_1 is algebraic over K $\Rightarrow$ we are done.

Case 2 v_1 is not algebraic over $K \Rightarrow v_i$ are algebraic over $K(v_1) \simeq K(x)$.

over $K(v_1) \cong K(x)$.

$v_i^{n_i} + a_{i1} v_i^{n_i-1} + \dots + a_{in_i} = 0 \qquad a_{ij} \in K(v_1)$

Let a = common denominator of all a_{ij}

$(av_i)^{n_i} + a \cdot a_{i1} (av_i)^{n_i-1} + \dots + a^{n_i} \cdot a_{in_i} = 0$

$\Rightarrow av_i$ is algebraic over $K[v_1]$.

$z \in L \Rightarrow z$ is a polynomial in $v_1 \dots v_n$

$\Rightarrow a^N z$ is a polynomial in $v_1, av_2, \dots av_n$

$\Rightarrow a^N z$ is algebraic over $K[v_1]$.

Pick $z = \frac{1}{c}$, $c \in K[v_1]$ coprime to a — can do this since infinitely many irred. polynomial in $K[x]$

$\frac{a^N}{c} \in K(v_1)$ algebraic over $K[v_1]$

(by Lemma) $\Rightarrow c = 1$ contradiction.

$\boxtimes$

<u>Thm 1</u> $\mathbb{K}$ alg. closed, all maximal ideals in $\mathbb{K}[x_1 \dots x_n]$ are $(x_1 - a_1, \dots x_n - a_n)$

<u>Proof</u>: $\mathfrak{m}$ = maximal ideal

$L = \mathbb{K}[x_1 \dots x_n]/I$ = field

By construction, L is finitely generated (by $\bar{x}_i$)

$\Rightarrow$ by Thm every element of L is algebraic over $\mathbb{K}$.

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But $\mathbb{K}$ is algebraically closed $\Rightarrow$ $\boxed{L = \mathbb{K}.}$

In particular, $x_i = a_i$ in L for some $a_i \in \mathbb{K}$

$\Rightarrow$ $x_i - a_i \in I \Rightarrow (x_1 - a_1, \dots, x_n - a_n) \subset I$.

Since $(x_1 - a_1, \dots, x_n - a_n)$ is maximal,

we are done. $\blacksquare$