Consequences of Nullstellensatz

Y = subset of $\mathbb{A}^n$, assume $\mathbb{K}$ alg. closed

Def $I(Y) = \{ f \in \mathbb{K}[x_1, \dots x_n] \mid f(x_1, \dots x_n) = 0$ for all points $(x_1, \dots x_n) \in Y \}$.

Lemma $I(Y)$ is a radical ideal

Proof • $f \in I(Y)$, g arbitrary. Then for all $(x_1, \dots x_n) \in Y$ we have $f(x_1, \dots x_n) g(x_1, \dots x_n) = 0$ so $fg \in I(Y)$.

• $f_1, f_2 \in I(Y) \Rightarrow$ for all $(x_1, \dots x_n) \in Y$ we have $f_1(x_1, \dots x_n) + f_2(x_1, \dots x_n) = 0 + 0 = 0 \Rightarrow f_1 + f_2 \in I(Y)$.

• Suppose $f^N \in I(Y)$ then f^N vanishes at all points of $Y \Rightarrow f$ vanishes at all points of Y
$\Rightarrow f \in I(Y)$. ☒

Lemma (a) If $Y_1 \subset Y_2$ then $I(Y_1) \supset I(Y_2)$

(b) $I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2)$

(c) Suppose J is an ideal in $\mathbb{K}[x_1, \dots x_n]$
Then $\boxed{I(Z(J)) = \sqrt{J}.}$

In particular, if J is radical then $I(Z(J)) = J$.

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__Proof__ (a) If $f(P) = 0$ for all $P \in Y_2$ then $f(P) = 0$ for all $P \in Y_1 \Rightarrow I(Y_1) \supset I(Y_2)$.

(b) f vanishes on $Y_1 \cup Y_2$ iff f vanishes at all points of Y_1 <u>and</u> f vanishes at all points of Y_2.

So $I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2)$.

(c) This is Nullstellensatz.

__Def__ Given $Y \subset K[x_1 \ldots x_n]$, we define $\overline{Y}$ as the smallest algebraic set containing Y.

__Lemma__ (a) We have $I(\overline{Y}) = I(Y)$.

(b) For any Y, we have $Z(I(Y)) = \overline{Y}$.

__Proof__ (a) $Y \subset \overline{Y} \Rightarrow I(\overline{Y}) \subset I(Y)$. Conversely, suppose $f \in I(Y)$. Then $\{f = 0\}$ is an algebraic set containing Y. <u>Recall</u> that the intersection of two algebraic sets is algebraic, so $\{f = 0\} \cap \overline{Y}$ is an algebraic set containing $Y \Rightarrow$ by def. of $\overline{Y}$ we get $\{f = 0\} \cap \overline{Y} = \overline{Y}$ and $\overline{Y} \subset \{f = 0\}$.

Therefore f vanishes at all points of $\overline{Y}$.

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(b) By definition, $Z(I(Y))$ is an algebraic set containing $Y \Rightarrow Z(I(Y)) \supset \overline{Y}$. Conversely, since $\overline{Y}$ is algebraic we can write $\overline{Y} = Z(J)$.

Then $I(Y) \underset{(a)}{=} I(\overline{Y}) = I(Z(J)) = \sqrt{J}$

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and $Z(I(Y)) = Z(\sqrt{J}) = Z(J) = \overline{Y}$. $\square$

We can combine the above into (recall $\mathbb{K}$ alg. closed)

Thm There is an inclusion-reversing bijection

$$\left\{ \begin{matrix} \text{algebraic subsets} \\ \text{of } \mathbb{A}^n \end{matrix} \right\} \underset{Z}{\overset{I}{\rightleftarrows}} \left\{ \begin{matrix} \text{radical ideals} \\ \text{in } \mathbb{K}[x_1 \ldots x_n] \end{matrix} \right\}$$

Rmk Later we will learn that there is a similar bijection for non-radical ideals, with more subtle "non-reduced scheme structure" on LHS.

Ex $n=1$ Ideals in $\mathbb{K}[x] \longleftrightarrow (f(x))$.

$\mathbb{K}$ alg. closed $\Rightarrow f(x) = (x-x_1) \ldots (x-x_d)$ (if $f \neq 0$, $\deg f > 0$)

$Z((f(x))) = \{x_1 \ldots x_d\}$ = finite subset of $\mathbb{A}^1$

So algebraic subsets of $\mathbb{A}^1$ are all finite sets, $\emptyset$ and the whole $\mathbb{A}^1$.

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Now we need to talk about prime ideals.

Def $p \underset{\text{ideal}}{\subset} \mathbb{K}[x_1 \dots x_n]$ is prime if whenever $fg \in p$ we have either $f \in p$ or $g \in p$.

Lemma (a) If p is prime then p is radical

(b) If p is prime, $p = I \cap J$ then $p = I$ or $p = J$.

Proof. (a) If $f^N \in p$ then $f \cdot f^{N-1} \in p \Rightarrow$ either $f \in p$ or $f^{N-1} \in p \Rightarrow$ by induction $f \in p$.

(b) Suppose $p \neq I$, $p \neq J$, then we can choose $f \in I$, $g \in J$ such that $f, g \notin p$. Now $fg \in I$, $fg \in J \Rightarrow$ $fg \in I \cap J = p \Rightarrow$ by def. of prime ideal $f \in p$ or $g \in p$. Contradiction. ☒

Def Y = alg. subset in $\mathbb{A}^n$ is called irreducible if it cannot be written as the union $Y = Y_1 \cup Y_2$ where Y_1, Y_2 are proper algebraic subsets of Y.

Thm Y is irreducible iff $I(Y)$ is prime.

Proof: $\Leftarrow$) Assume $I(Y)$ is prime, and $Y = Y_1 \cup Y_2$. Then $I(Y) = I(Y_1) \cap I(Y_2) \Rightarrow$ by lemma either

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$I(Y_1) = I(Y)$ or $I(Y_2) = I(Y) \Rightarrow$ by applying Z

we get $Y_1 = Y$ or $Y_2 = Y$. So Y is irreducible.

2) Assume Y is irreducible, and $fg \in I(Y)$.

Then $Y \subset (\{fg = 0\}) = (\{f=0\} \cup \{g=0\})$

Define $Y_1 = Y \cap \{f=0\}$, $Y_2 = Y \cap \{g=0\}$, then $Y = Y_1 \cup Y_2$

Since Y is irreducible, we get $Y_1 = Y$ or $Y_2 = Y$

$\Rightarrow$ either $Y \subset \{f=0\}$ or $Y \subset \{g=0\} \Rightarrow$ either

$f \in I(Y)$ or $g \in I(Y)$. So $I(Y)$ is prime. ∎