Noetherian rings

Def R = ring, it is Noetherian if any ascending (increasing) chain of ideals

$$I_1 \subset I_2 \subset I_3 \subset \dots$$

eventually stabilizes, that is, there exists N such that $I_k = I_{k+1}$ for $k \geq N$.

Lemma R is Noetherian $\iff$ any ideal in I is finitely generated, that is, $I = (f_1 \dots f_k)$.

Proof: (1) Assume R is Noetherian, and I is an ideal. Define a sequence of polynomials

$$f_1 \in I, \quad f_2 \in I \setminus (f_1), \quad f_3 \in I \setminus (f_1, f_2) \dots$$

if $(f_1 \dots f_k) = I$, we stop, otherwise find

$$f_{k+1} \in I \setminus (f_1 \dots f_k) \text{ and proceed.}$$

If we never stop, get a sequence of ideals

$$(f_1) \subsetneq (f_1, f_2) \subsetneq (f_1, f_2, f_3) \subsetneq \dots$$

By Noetherian condition, it stops. Contradiction $\Rightarrow$

$$I = (f_1 \dots f_k) \text{ for some } k.$$

(2) Assume any ideal is finitely generated, let us prove R is Noetherian. Suppose

R is Noetherian. Suppose

$$I_1 \subset I_2 \subset I_3 \subset \dots \text{ is an infinite chain of ideals.}$$

Define $I = \bigcup_{k=1}^{\infty} I_k$, let us prove it is an ideal.

- $f \in I \Rightarrow f \in I_k$ for some $k \Rightarrow fg \in I_k \subset I$.
  g any
- $f_1 \in I_k$, $f_2 \in I_l \Rightarrow$ if $k \le l$ then $f_1 \in I_l$ and $f_1 + f_2 \in I_l \subset I$.

By assumption, $I = (f_1 \dots f_m)$ and

$$f_1 \in I_{k_1},\ f_2 \in I_{k_2},\ \dots\ f_m \in I_{k_m}$$

$$\Rightarrow \text{all } f_i \in I_{\max(k_1 \dots k_m)} \Rightarrow \underset{I}{\underset{\parallel}{(f_1 \dots f_m)}} \subset I_{\max(k_1 \dots k_m)}$$

Therefore $I = I_{\max}$ and we are done. ∎

Thm If R is Noetherian then $R[x]$ is Noetherian.

Pf Let $I \subset R[x]$ be an ideal, we need to prove it is finitely generated. For $f \in I$ we can write

$$f = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0 \qquad a_i \in R,\ a_n \neq 0$$

Define $a_{top}(f) = a_n$ and $\deg(f) = n$.

Define a sequence of polynomials in I:

- $f_1 =$ polynomial of smallest degree in I
- $f_2 =$ polynomial of smallest degree in $I \setminus (f_1)$
- ... $f_k =$ polynomial of smallest degree in $I \setminus (f_1 \dots f_k)$

If $(f_1 \dots f_k) = I$ we stop and I is finitely generated

f_k = polynomial of smallest degree in $I - (f_1 \ldots f_{k-1})$

If $(f_1 \ldots f_k) = I$, we stop and I is finitely generated.
Otherwise consider the sequence of ideals

$$(a_{top}(f_1), \ldots a_{top}(f_k)) \subset R.$$

Since R is noetherian then at some point these stabilize, so $a_{top}(f_m) \in (a_{top}(f_1) \ldots a_{top}(f_k))$ for $m \geq k$.
Write $a_{top}(f_m) = \sum_{i=1}^{k} u_i \, a_{top}(f_i)$, then for

$$g = f_m - \sum u_i \, x^{\deg(f_m) - \deg(f_i)} f_i$$

the top coefficient cancels out and
$\deg(g) < \deg(f_m)$, so by our assumption
$g \in (f_1 \ldots f_k) \Rightarrow f_m \in (f_1 \ldots f_k)$. ∎

<u>Corollary</u> (Hilbert Basis Theorem) $\mathbb{K}$ = field
$\mathbb{K}[x_1 \ldots x_n]$ is Noetherian, so any ideal in $\mathbb{K}[x_1 \ldots x_n]$ is finitely generated.

<u>Corollary</u> Z = algebraic set in $\mathbb{A}^n \Rightarrow Z$ is defined by finitely many equations $\{f_1 = \ldots = f_k = 0\}$.

<u>Proof</u>: $I(Z) = (f_1, \ldots f_k)$ and $Z = Z(I(Z)) = Z(\underbrace{f_1 \ldots f_k}_{\text{ideal}})$

$$= Z(\underbrace{\{f_1 \ldots f_k\}}_{\text{finite set}}).$$ ∎

$= Z(\{f_1, \dots, f_k\})$.
finite set. ∎

Corollary: $Z_1 \supset Z_2 \supset Z_3 \supset \dots$ descending chain of algebraic subsets $\Rightarrow$ Z_i stabilize.

Proof We have $I(Z_1) \subset I(Z_2) \subset \dots \subset$ which stabilize by Noetherian property. ∎

Recall that Z is called irreducible if

$$Z = Z_1 \cup Z_2, \quad Z_1, Z_2 \text{ algebraic} \Rightarrow Z_1 = Z \text{ or } Z_2 = Z.$$

Thm If Z = algebraic set, then we can uniquely write $Z = Z_1 \cup \dots \cup Z_k$ where Z_i are irreducible.

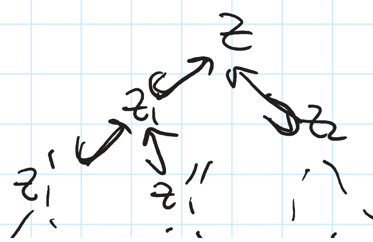
finitely many

Such Z_i are called irreducible components of Z.

Proof If Z is irreducible, we are done. Otherwise there exist Z_1 and Z_2 such that $Z = Z_1 \cup Z_2$, and $Z_1 \neq Z$, $Z_2 \neq Z$. If Z_1, Z_2 are irreducible then we are done, otherwise we can write $Z_1 = Z_1' \cup Z_1''$ and so on.

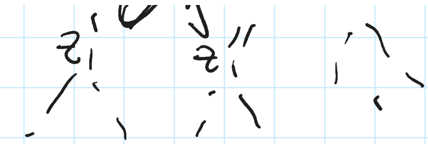
$Z = Z_1' \cup Z_1'' \cup Z_2$

We can draw this as a binary tree:



where all inclusions are proper.

Each chain of proper subsets



Each chain of proper subsets eventually stops (Noetherian!), the tree is actually finite, and all leaves are irreducible.

Therefore Z = union of finitely many irreducible subsets. ∎

Cor I = radical ideal in $\mathbb{K}[x_1 \ldots x_n]$ (primary decomposition)

$$\Rightarrow I = p_1 \cap p_2 \cap \ldots \cap p_k$$

where p_i are prime ideals.

Proof $Z(I) = Z_1 \cup \ldots \cup Z_k$, define $p_i = I(Z_i)$

Z_i are irreducible $\Leftrightarrow$ p_i are prime. ∎

Lemma $Z = Z_1 \cup \ldots \cup Z_k$, Z_i irreducible

Assume $Z = A \cup B$ where A, B are algebraic sets.

Then for all i either $Z_i \subset A$ or $Z_i \subset B$, so that

A = (union of some Z_i) $\quad$ B = (union of some Z_i).

Proof $Z_i = (A \cup B) \cap Z_i = (A \cap Z_i) \cup (B \cup Z_i)$.

Since Z_i is irreducible, either $A \cap Z_i = Z_i$ so $Z_i \subset A$

or $B \cap Z_i = Z_i$ so $Z_i \subset B$. ∎

Cor The irreducible components are uniquely determined by Z.