Lecture 6 (1/22)

Sunday, January 21, 2024 2:03 AM

Corollary (Hilbert Basis Theorem) $\mathbb{K}$ = field

$\mathbb{K}[x_1 \dots x_n]$ is Noetherian, so any ideal in $\mathbb{K}[x_1 \dots x_n]$ is finitely generated.

Corollary Z = algebraic set in $\mathbb{A}^n \Rightarrow Z$ is defined by finitely many equations $\{f_1 = \dots = f_k = 0\}$.

Proof: $I(Z) = (f_1, \dots f_k)$ and $Z = Z(I(Z)) = Z(\underbrace{f_1 \dots f_k}_{\text{ideal}})$

$= Z(\underbrace{\{f_1 \dots f_k\}}_{\text{finite set}})$. $\square$

Corollary: $Z_1 \supset Z_2 \supset Z_3 \supset \dots$ descending chain of algebraic subsets $\Rightarrow Z_i$ stabilize.

Proof We have $I(Z_1) \subset I(Z_2) \subset \dots \subset$ which stabilize by Noetherian property. $\square$

Recall that Z is called irreducible if

$Z = Z_1 \cup Z_2$, Z_1, Z_2 algebraic $\Rightarrow Z_1 = Z$ or $Z_2 = Z$.

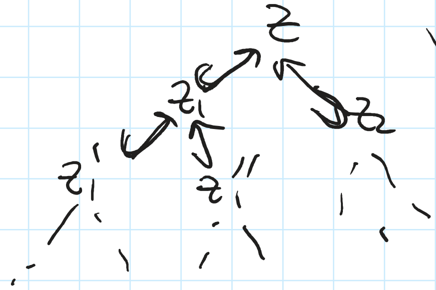
Thm If Z = algebraic set, then we can uniquely write $Z = \underbrace{Z_1 \cup \dots \cup Z_k}_{\text{finitely many}}$ where Z_i are irreducible.

Such Z_i are called irreducible components of Z

Such Z_i are called <u>irreducible components</u> of Z.

<u>Proof</u> If Z is irreducible, we are done. Otherwise there exist Z_1 and Z_2 such that $Z = Z_1 \cup Z_2$, and $Z_1 \neq Z$, $Z_2 \neq Z$. If Z_1, Z_2 are irreducible then we are done, otherwise we can write $Z_1 = Z_1' \cup Z_1''$ and so on. $Z = Z_1' \cup Z_1'' \cup Z_2$

We can draw this as a binary tree:



where all inclusions are proper. Each chain of proper subsets eventually stops, so, the tree (Noetherian!) is actually finite, and all leaves are irreducible.

Therefore Z = union of finitely many irreducible subsets. ∎

<u>Cor</u> I = radical ideal in $K[x_1 \ldots x_n]$

$$\Rightarrow I = p_1 \cap p_2 \cap \ldots \cap p_k$$

(primary decomposition)

where p_i are prime ideals.

<u>Proof</u> $Z(I) = Z_1 \cup \ldots \cup Z_k$, define $p_i = I(Z_i)$

Z_i are irreducible $\Leftrightarrow$ p_i are prime. ∎

<u>Lemma</u> $Z = Z_1 \cup \ldots \cup Z_k$, Z_i irreducible

Assume $Z = A \cup B$ where A, B are algebraic sets.

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Then for all i either $Z_i \subset A$ or $Z_i \subset B$, so that

$$A = (\text{union of some } Z_i) \qquad B = (\text{union of some } Z_i).$$

<u>Proof</u> $Z_i = (A \cup B) \cap Z_i = (A \cap Z_i) \cup (B \cup Z_i)$.

Since Z_i is irreducible, either $A \cap Z_i = Z_i$ so $Z_i \subset A$

or $B \cap Z_i = Z_i$ so $Z_i \subset B$.

<u>Cor</u> the irreducible components are uniquely determined by Z.

<u>Zariski topology</u>

① Zariski topology on $\mathbb{A}^n$

We define a structure of a topological space on $\mathbb{A}^n$ using algebraic geometry. It is very strange, but useful to work with, and is defined over any field.

First we recall the basic defs:

Def A top. space X is a set with a collection of open subsets $U \subset X$, satisfying the following axioms:

① $\emptyset$ and X are open

② U_1, U_2 open $\Rightarrow U_1 \cap U_2$ open (more generally, finite intersections of open sets are open).

③ U_α, $\alpha \in A$ open $\Rightarrow \bigcup_{\alpha \in A} U_\alpha$ open (arbitrary unions of open sets are open).

We say that $Z \subset X$ is closed if $X \setminus Z$ is open.

1a $\emptyset$ and Z are closed

2a Z_1, Z_2 closed $\Rightarrow Z_1 \cup Z_2$ open (finite union of closed are closed).

3a Z_α, $\alpha \in A$ closed $\Rightarrow \bigcap_{\alpha \in A} Z_\alpha$ closed (arbitrary intersections of closed are closed).

Def The Zariski topology on $\mathbb{A}^n$ is defined as

Def The <u>Zariski topology</u> on $\mathbb{A}^n$ is defined as follows:

- All algebraic sets in $\mathbb{A}^n$ are <u>closed</u>.
- U is open if $\mathbb{A}^n \setminus U$ is closed $\Leftrightarrow$ $\mathbb{A}^n \setminus U$ is an algebraic set.

Lemma a) This defines a topology on $\mathbb{A}^n$

b) Over $\mathbb{R} \setminus \mathbb{C}$, this is a coarsening of "classical"/"standard" topology. That is:

- Z is closed in Zariski top. $\Rightarrow$ closed in "standard" top
- U is open in Zariski top $\Rightarrow$ open in "standard" top.

Proof (a) We actually know this:

- $\emptyset$ and $\mathbb{A}^n$ are alg. sets
- Z_1, Z_2 algebraic sets $\Rightarrow Z_1 \cup Z_2$ algebraic
- Z_α, $\alpha \in A$ algebraic $\Rightarrow \bigcap_{\alpha \in A} Z_\alpha$ algebraic.

So algebraic sets satisfy axioms for closed subsets.

(b) If $f(x_1 \dots x_n)$ is a polynomial over $\mathbb{R}/\mathbb{C}$ then it is continuous in "standard" topology

$\Rightarrow \{f(x_1 \dots x_n) = 0\}$ is closed in "standard" topology.

So $Z(T)$ is closed in "standard" topology for any set T

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<u>Ex</u> In $\mathbb{A}^1$, all closed subsets are

$\{\emptyset, \mathbb{A}^1, \text{all finite sets in } \mathbb{A}^1\}$

All open subsets: $\{\emptyset, \mathbb{A}^1, \text{all complements to finite sets in } \mathbb{A}^1\}$

In particular, it is non-Hausdorff (if $\mathbb{k}$ is infinite): any two non-empty open subsets have a non-empty intersection.

<u>Note</u>: The Zariski topology on $\mathbb{A}^2$ is <u>not</u> the product topology of those on $\mathbb{A}^1$

<u>Proof</u>: In product topology on $\mathbb{A}^2$, open subsets are

$\emptyset, \mathbb{A}^2$ and (complement to finite set) $\times$ (complement to finite set)

There are many more open subsets, in particular $\{x_1 \neq x_2\}$.