Thm Assume $\mathbb{K}$ alg. closed. Then $\mathbb{A}^n$ is irreducible.

Proof $\mathbb{A}^n = Z(0)$, so by Nullstellensatz $I(\mathbb{A}^n) = \sqrt{0} = 0$. Since 0 is a prime ideal (prove it!), $\mathbb{A}^n$ is irreducible. ∎

Warning: If $\mathbb{K}$ is a finite field then $\mathbb{A}^n$ is a finite set $\Rightarrow$ union of points $\Rightarrow$ reducible.

Warning: If $\mathbb{K}$ is a finite field then $I(\mathbb{A}^n) \neq 0$.
For example, $f(x) = \prod_{a \in \mathbb{K}} (x-a)$ is a nonzero polynomial of degree $|\mathbb{K}|$ which vanishes at all points of $\mathbb{K}$.

Lemma Assume $\mathbb{K}$ is an infinite field (not necessarily alg. closed, for example, $\mathbb{R}$). Then $\mathbb{A}^n(\mathbb{K})$ is irreducible.

Proof: For $n=1$ this is an exercise. For general n, assume $\mathbb{A}^n = Z_1 \cup Z_2$, Z_i closed and proper. Let $P \in Z_1 \setminus Z_2$ and $Q \in Z_2 \setminus Z_1$, consider the line ℓ through PQ. Then:

- $\ell = (\ell \cap Z_1) \cup (\ell \cap Z_2)$, and both are alg. sets
- $\ell \cap Z_1 \neq \ell$ since it does not contain Q
- $\ell \cap Z_2 \neq \ell$ since it does not contain P.

But $\ell = \mathbb{A}^1$ is irreducible, contradiction. ∎

Cor Assume K is alg. closed. Then any nonempty Zariski open subset of $\mathbb{A}^n$ is dense.

Proof Assume U is open, nonempty, Z is closed and $Z \supset U$. We need to prove $Z = \mathbb{A}^n$.

Denote $W = \mathbb{A}^n \setminus U$, it is closed. Observe

$\mathbb{A}^n = Z \cup W$ (if a point is in U it is in Z, otherwise in W)

Since $U \neq \emptyset$ we have $W \neq \mathbb{A}^n$. Since $\mathbb{A}^n$ is irreducible, we get $Z = \mathbb{A}^n$.

$\boxtimes$

Lemma a) If $Z_1 \subset \mathbb{A}^n$ and $Z_2 \subset \mathbb{A}^m$ are closed then $Z_1 \times Z_2$ are closed in $\mathbb{A}^{m+n}$

b) If $U_1 \subset \mathbb{A}^n$ and $U_2 \subset \mathbb{A}^m$ are open then $U_1 \times U_2$ are open in $\mathbb{A}^{m+n}$

c) The diagonal $\Delta \subset \mathbb{A}^n \times \mathbb{A}^n$ is closed

Proof a) Z_1 closed in $\mathbb{A}^n$ $\Rightarrow$ $Z_1 \times \mathbb{A}^m$ is closed in $\mathbb{A}^{m+n}$ (use same equations)

$$Z_1 \times Z_2 = (Z_1 \times \mathbb{A}^m) \cap (\mathbb{A}^n \times Z_2)$$
closed.

b) $U_1 = \mathbb{A}^n \setminus Z_1$, $U_2 = \mathbb{A}^m \setminus Z_2$

$$\mathbb{A}^{m+n} \setminus (U_1 \times U_2) = \underbrace{(Z_1 \times \mathbb{A}^m) \cup (\mathbb{A}^n \times Z_2)}_{\text{closed}}.$$

$\mathbb{A}^{m+n} \setminus (U_1 \times U_2) = (Z_1 \times \mathbb{A}^m) \cup (\mathbb{A}^n \times Z_2)$.
closed

c) $\Delta = \{x_1 = x_{n+1}, x_2 = x_{n+2}, \dots x_n = x_{2n}\}$.
alg. set in $\mathbb{A}^{2n}$.

Def A principal open subset $D(f)$ is

$$D(f) = \{f \neq 0\} \subset \mathbb{A}^n.$$

Clearly, $D(f)$ is open since $\mathbb{A}^n \setminus D(f) = \{f = 0\}$ closed.

Lemma Any open subset is a finite union of $D(f)$.

In particular, $D(f)$ form a base of Zariski topology.

Proof Let $T \subset \mathbb{K}[x_1 \dots x_n]$ any subset, I = ideal gen. by T

$Z = Z(T)$ = alg. set. Recall:

Noetherian $\mathbb{K}[x_1, \dots x_n]$

$Z = Z(T) = Z(I)$ and $I = (f_1 \dots f_k)$ finite

Therefore $Z = \{f_1 = 0\} \cap \{f_2 = 0\} \cap \dots \cap \{f_k = 0\}$

and $\mathbb{A}^n \setminus Z = D(f_1) \cup D(f_2) \cup \dots \cup D(f_k)$

$= \{f_1 \neq 0\} \cup \{f_2 \neq 0\} \cup \dots \cup \{f_k \neq 0\}$. $\square$

Def $X \subset \mathbb{A}^n$ any subset

We define Zariski topology on X as the induced topology:

$\{\text{closed}\} = X \cap Z$, $Z \subset \mathbb{A}^n$ closed

$\{\text{closed}\} = X \cap Z$, $Z \subset \mathbb{A}^n$ closed
$\{\text{open}\} = X \cap U$, $U \subset \mathbb{A}^n$ open

Rmk If X is algebraic set in $\mathbb{A}^n$, then

$$\left\{\begin{matrix}\text{closed subsets}\\ \text{of } X\end{matrix}\right\} = \left\{\begin{matrix}\text{algebraic}\\ \text{sets contained in } X\end{matrix}\right\}.$$

Pf: ① If $Z \subset X$ algebraic, then Z is closed in $\mathbb{A}^n$ and $Z = Z \cap X$.

② If $Z \subset \mathbb{A}^n$ closed, then Z is algebraic and $Z \cap X$ algebraic. (since X was algebraic).

Zariski topology on $\mathbb{P}^n$

$$\{\text{closed sets}\} = \left\{\begin{matrix}\text{projective}\\ \text{algebraic sets}\end{matrix}\right\} = Z(I)$$

I = homogeneous ideal in $\mathbb{K}[x_0, \dots, x_n]$

Homogeneous coords on $\mathbb{P}^n$:
$[x_0 : \dots : x_n]$

Lemma This is a topology. **Pf**: Exercise.

Ex $U_i = \{x_i \neq 0\} \cong \mathbb{A}^n$ open $\quad Z_i = \{x_i = 0\} \simeq \mathbb{P}^{n-1}$ closed

Lemma U_i / Z_i with induced Zariski topology $= \mathbb{A}^n / \mathbb{P}^{n-1}$ with Zariski topology

$= \mathbb{A}^n / \mathbb{P}^{n-1}$ with Zariski topology.

Proof: Next time.