Recall A top. space X is irreducible if $X = Z_1 \cup Z_2$, Z_i closed $\Rightarrow Z_1 = X$ or $Z_2 = X$.

Lemma X = irreducible top. space, $U \subset X$ open, $U \neq \emptyset$. Then U is dense in X and irreducible.

Proof: (1) Suppose $U \subset Z$, Z closed, $W = X \setminus U$. Then $X = Z \cup W$. Since $U \neq \emptyset$, $W \neq X \Rightarrow Z = X$.
So $\overline{U} = X$ and U is dense.

(2) Suppose $U = Z_1' \cup Z_2'$ where Z_i' closed in U. Then $Z_i' = Z_i \cap U$ for some closed $Z_i \subset X$ and $Z_1 \cup Z_2 \supset U$. Since U is dense, we get $Z_1 \cup Z_2 = X$. Since X is irreducible, we get $Z_1 = X \Rightarrow Z_1' = U$
or $Z_2 = X \Rightarrow Z_2' = U$.

Cor $\mathbb{A}^{n+1} \setminus \{0\}$ is irreducible $\Rightarrow \mathbb{P}^n$ is irreducible.

Lemma $U_i = \{x_i \neq 0\} \subset \mathbb{P}^n$. Then induced topology on U_i agrees with Zariski topology on $U_i \simeq \mathbb{A}^n$.

Pf (1) Assume $Z = \{F_k = 0\}$ closed in $\mathbb{P}^n$, where F_i are homogeneous. Then $Z \cap U_i = \{F_k(x_0, \ldots x_i \ldots x_n) = 0,\ x_i \neq 0\}$

homogeneous. Then $Z \cap U_i = \{F_k(x_0, \dots x_i \dots x_n) = 0, x_i \neq 0\}$

$$= \{F_k(\frac{x_0}{x_i}, \dots, 1, \dots \frac{x_n}{x_i}) = 0\} =$$

$$= \{F_k(y_0, \dots 1, \dots y_n) = 0 \text{ where } y_j = \frac{x_j}{x_i}\}.$$

coord on $\mathbb{A}^n$.

So $Z \cap U_i$ is closed in $\mathbb{A}^n$.

(2) Assume $Z \subset U_i$ closed, let $\overline{Z}$ be the projective closure (obtained by homogenization of equations of Z, see HW1). Then $\overline{Z}$ is closed in $\mathbb{P}^n$, and $\overline{Z} \cap U_i = Z$ so Z is closed in induced topology.

__Thm__ (projective Nullstellensatz) Let J be a homogeneous ideal in $\mathbb{K}[x_0, \dots x_n]$, $\mathbb{K}$ alg. closed. Then $Z(J) \subset \mathbb{P}^n$ is empty iff J contains $(x_0, \dots x_n)^N$ for some N.

__Proof__: (1) Assume J contains $(x_0, \dots x_n)^N$, then $Z(J)$ satisfies $x_0^N = x_1^N = \dots = x_n^N = 0 \Rightarrow$

$x_0 = \dots = x_n = 0$, contradiction, so $Z(J) = \emptyset$.

(2) Assume $Z(J) = \emptyset$, then the corresponding subset $\tilde{Z} \subset \mathbb{A}^{n+1}$ is either empty or $\{0\}$.

__Case 1__ $\tilde{Z} = \emptyset \Rightarrow$ by Nullstellensatz $J = \mathbb{K}[x_0 \dots x_n]$

__Case 2__ $\tilde{Z} = \{0\} \Rightarrow$ by Nullstellensatz

$$\sqrt{J} = I(\{0\}) = (x_0, \dots x_n)$$

$$\sqrt{J} = I(\{0\}) = (x_0, \ldots x_n)$$

So $x_0^N, \ldots x_n^N \in J$ and we are done. ☒

Rings of functions Let X = alg. subset of $\mathbb{A}^n$.

Def A function $f : X \to \mathbb{K}$ is regular if there is $\tilde{f} \in \mathbb{K}[x_1 \ldots x_n]$ such that $\tilde{f}|_X = f$.

$A(X)$ = ring of regular functions on X

We have surjective algebra homomorphism restriction $\mathbb{K}[x_1 \ldots x_n] \xrightarrow{\varphi} A(X)$

$$\operatorname{Ker} \varphi = I(X)$$

By Isomorphism theorem $\boxed{A(X) = \dfrac{\mathbb{K}[x_1 \ldots x_n]}{I(X)}}$

Lemma a) There is a bijection $\{\text{ideals in } A(X)\}$ $\updownarrow$ $\{\text{ideals in } \mathbb{K}[x_1 \ldots x_n] \text{ containing } I(X)\}$

b) Maximal ideals in $A(X)$ $=$ Maximal ideals in $\mathbb{K}[x_1 \ldots x_n]$ containing $I(X)$ $\underset{(*)}{=}$ points in X

(*) if $\mathbb{K}$ alg. closed.

c) $A(X)$ is Noetherian.

Proof a) • Let J be an ideal in $A(X)$, then $\varphi^{-1}(J)$

Proof (a): Let J be an ideal in $A(X)$, then $\varphi^{-1}(J)$

is an ideal in $\mathbb{K}[x_1 \dots x_n]$ (prove it!) containing $\underset{\underset{I(X)}{\|}}{\operatorname{Ker}\varphi}$.

- Conversely, if $\tilde{J}$ ideal in $\mathbb{K}[x_1 \dots x_n]$ and

  $\operatorname{Ker}\varphi \subseteq \tilde{J}$ then $\varphi(\tilde{J})$ is an ideal in $A(X)$.

Indeed, if $f \in \tilde{J}$ and $g \in A(X)$ then there exist $\tilde{g}$ such that $\varphi(\tilde{g}) = g$. Then   Note: φ surjective!

$$\varphi(f) \cdot g = \varphi(f) \cdot \varphi(\tilde{g}) = \varphi(f\tilde{g}) \in \varphi(\tilde{J})$$

b) The first part is clear from (a). For the second part, by Nullstellensatz we get $m = (x_1 - a_1, \dots x_n - a_n)$ and $m \supset I(X)$ iff $X \supset Z(m) = \{(a_1, \dots a_n)\}$.

c) Clear from (a). $\blacksquare$