Recall: X = alg. set in $\mathbb{A}^n$

$$A(X) = \frac{\mathbb{K}[x_1, \ldots x_n]}{I(X)}$$

$$\left\{ \begin{matrix} \text{ideals in} \\ A(X) \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{ideals in } \mathbb{K}[x_1 \ldots x_n] \\ \text{containing } I(X) \end{matrix} \right\}$$

Cor Closed subsets in $X \longleftrightarrow$ ideals in $A(X)$.

Lemma Assume $f \in A(X)$ and $f \neq 0$ on X. Then f is invertible in $A(X)$. ($\mathbb{K}$ alg. closed)

Proof Let $J = (f, I(X))$. Then

$Z(J) = \{f = 0\} \cap X = \emptyset$, so by Nullstellensatz

$J = \mathbb{K}[x_1, \ldots x_n] \Rightarrow \alpha f + g = 1$ for $\alpha \in \mathbb{K}[x_1 \ldots x_n]$, $g \in I(X)$.

Then $\varphi(\alpha) \cdot f = 1$ in $A(X)$.

Morphisms $X \subset \mathbb{A}^n_{x_1 \ldots x_n}$, $Y \subset \mathbb{A}^m_{y_1 \ldots y_m}$ alg. sets. (affine)

Def A *morphism* $\varphi : X \longrightarrow Y$ is a function such that there exist polynomials $p_1(x_1 \ldots x_n), \ldots p_m(x_1 \ldots x_n)$ such that $\varphi(x_1, \ldots x_n) = (p_1(x_1 \ldots x_n), \ldots p_m(x_1 \ldots x_n))$

Ex A morphism $X \longrightarrow \mathbb{A}^1$ is a regular function on X.

Ex A morphism $X \to \mathbb{A}^1$ is a regular function on X.

Ex $X = \mathbb{A}^1_t$, $Y = \{x^2 = y^3\} \subset \mathbb{A}^2$

$$\varphi(t) = (t^3, t^2)$$

Well defined since $\varphi(t) \in Y$ for all t:

$$(t^3)^2 = (t^2)^3.$$

Thm There is a bijection

$$\left\{ \begin{matrix} \text{morphisms} \\ X \xrightarrow{\varphi} Y \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{algebra homomorphisms} \\ A(Y) \xrightarrow{\varphi^*} A(X) \end{matrix} \right\}$$

Pf (1) Suppose $\varphi : X \to Y$ is a morphism, define $\varphi^* : A(Y) \to A(X)$ as follows: $\boxed{\varphi^* f = f \circ \varphi}$

For $f \in A(Y)$, find a polynomial $\tilde{f}(y_1 \ldots y_m)$ such that $\tilde{f}|_Y = f$. Then $f(\varphi(x_1 \ldots x_n)) = \tilde{f}(\varphi(x_1 \ldots x_n)) =$

$= \tilde{f}(p_1(x_1 \ldots x_n), \ldots, p_m(x_1 \ldots x_n)) =$ polynomial in x_i

Clearly, this is an algebra homomorphism.

(2) Suppose $\alpha : A(Y) \to A(X)$ homomorphism, let us reconstruct φ such that $\varphi^* = \alpha$. Note that $A(Y)$ is generated by $y_1 \ldots y_m$, consider $\alpha(y_1), \ldots \alpha(y_m) \in A(X)$.

By def., there exist polynomials $p_1, \ldots p_m$ such that $p_i = \alpha(y_i)$ on X, and define such that $p_i = \alpha(y_i)$ on X, and define

$$\varphi(x_1, \dots x_n) = (p_1(x_1 \dots x_n), \dots p_m(x_1 \dots x_n)).$$

By definition, φ is a morphism, and we need to check two things:

(a) If $(x_1 \dots x_n) \in X$ then $\varphi(x_1 \dots x_n) \in Y$. Indeed, suppose $g(y_1 \dots y_m) \in I(Y)$, then for $\bar{x} \in X$ we get

$$g(p_1(x_1 \dots x_n), \dots p_m(x_1 \dots x_n)) = g(\alpha(y_1), \dots \alpha(y_m)) =$$

$$\overset{\text{φ is a homomorphism}}{=} \alpha(g(y_1 \dots y_m)) = \alpha(0) = 0$$

in $A(X)$

(b) $\underline{\varphi^* = \alpha}$. Indeed, for $f \in A(Y)$

$$\varphi^* f = f(p_1(x_1 \dots x_n), \dots p_m(x_1 \dots x_n)) = f(\alpha(y_1) \dots \alpha(y_m))$$

$$= \alpha(f(y_1 \dots y_m)) \Rightarrow \varphi^* f = \alpha(f).$$

__Thm__ If $\varphi : X \to Y$ is a morphism then φ is continuous in Zariski topology (X, Y = affine alg. sets)

__Pf__: Suppose $Z \subset Y$ closed, we need to prove $\varphi^{-1}(Z)$ is closed in X. Note that:

$$\bar{x} \in \varphi^{-1}(Z) \Leftrightarrow \varphi(\bar{x}) \in Z \Leftrightarrow \text{for all } g \in I(Z)$$

we have $g(\varphi(\bar{x})) = 0 \Leftrightarrow \varphi^*(g)(\bar{x}) = 0$

So $\varphi^{-1}(Z)$ = alg. set defined by $\varphi^* I(Z)$

we have $g(\varphi(x)) = 0 \iff \varphi^*(g)(x) = 0$

So $\varphi^{-1}(Z)$ = alg. set defined by $\varphi^* I(Z)$.

not necessarily an ideal

Ex $X = \mathbb{A}^1 \longrightarrow Y = \{x^2 = y^5\}$ $\qquad \varphi^* x = t^5$

$\varphi(t) = (t^5, t^2) \qquad \varphi^* y = t^2$

$Z = \{(0,0)\} \subset Y \qquad I(Z) = (x, y) \subset A(Y) = \dfrac{K[x,y]}{(x^2 - y^5)}$

$$\varphi^* I(Z) = \mathrm{Span}\langle \underset{\varphi^* y}{t^2}, \underset{\varphi^* y^2}{t^4}, \underset{\varphi^* x}{t^5}, \underset{\varphi^* y^3}{t^6}, \underset{\varphi^*(xy)}{t^7}, \dots \rangle$$

Def X and Y are isomorphic if $\exists\, \varphi : X \to Y$ morphism φ = bijection and $\varphi^{-1} : Y \to X$ is a morphism.

Thm φ is an isomorphism $\iff \varphi^* : A(Y) \to A(X)$ is an algebra isomorphism.

Ex $W = \{u = v^2\} \subset \mathbb{A}^2_{u,v} \qquad Y = \{x^2 = y^3\} \subset \mathbb{A}^2_{x,y}$

$\varphi(u,v) = (uv, u)$

- Well defined: if $(u,v) \in W$ then $\varphi(u,v) \in Y$

  Indeed, $(uv)^2 = (v^2 \cdot v)^2 = v^6$

  $u^3 = (v^2)^3 = v^6$ ✓

- $A(W) = \underline{K[u,v]} \xleftarrow{\ \varphi^*\ } A(Y) = \underline{K[x,y]}$

- $A(W) = \dfrac{\mathbb{K}[u,v]}{(u-v^2)} \xleftarrow{\varphi^*} A(Y) = \dfrac{\mathbb{K}[x,y]}{(x^2-y^3)}$

$\varphi^*(x) = uv$ $\qquad$ $\varphi^*(x^2-y^3) = (uv)^2 - u^3 =$

$\varphi^*(y) = u$ $\qquad$ $= u^2(v^2-u) \in I(W)$

So $\varphi^*(x^2-y^3) = 0$ in $A(W)$ ✓