MAT 248A, Winter 2024
Homework 1

Due before 11:00 on Wednesday, January 17

Please write the homework solutions in connected sentences and explain your work. Mark the answers to each question. Scan or take pictures of your homework and upload it to Gradescope before due time.

In all problems we work over an arbitrary field $\mathbf{k}$ unless stated otherwise.

1. Recall that the projective space $\mathbb{P}^n$ is defined as the set of $(n+1)$-tuples $[x_0 : \ldots : x_n]$ such that not all x_i vanish, modulo equivalence relation $[x_0 : \ldots : x_n] \sim [\lambda x_0 : \ldots : \lambda x_n]$ for $\lambda \neq 0$.
a) Let $U_i = \{x_i \neq 0\}$. Prove that any point in U_i has a unique representative with $x_i = 1$, and that U_i is isomorphic to the affine space $\mathbb{A}^n$ with coordinates x_j/x_i for all $j \neq i$.
b) Prove that $\mathbb{P}^n \setminus U_i$ is an algebraic set in $\mathbb{P}^n$ isomorphic to $\mathbb{P}^{n-1}$.
c) Consider $Z = \{x_0^2 = x_1 x_2\} \subset \mathbb{P}^2$. Describe the intersections of Z with the charts U_0, U_1, U_2.

2. a) Let $f(x_1, \ldots, x_n)$ be a degree d polynomial. Prove that there is a unique homogeneous degree d polynomial $F(x_0, x_1, \ldots, x_n)$ such that $F(1, x_1, \ldots, x_n) = f(x_1, \ldots, x_n)$. Such F is usually called a homogenization of f.
b) Let Z be an algebraic set in $\mathbb{A}^n$. Use part (a) to constuct an algebraic set $\overline{Z}$ in $\mathbb{P}^n$ such that $\overline{Z} \cap U_0 = Z$. Such Z is called the **projective closure** of Z and the difference $\overline{Z} \setminus Z$ is called the "set of points of Z at infinity".
c) Find the projective closure and the set of points at infinity for the parabola $Z_1 = \{y = x^2\}$ and hyperbola $Z_2 = \{xy = 1\}$.
d) Assume $\mathbf{k} = \mathbb{C}$. Find the set of points at infinity for $Z = \{(x-a)^2 + (y-b)^2 = R^2\}$.

3. a) Prove that there is a unique line through any two distinct points in $\mathbb{A}^n$.
b) Prove that there is a unique line through any two distinct points in $\mathbb{P}^n$.
c) Prove that a line in $\mathbb{A}^n$ (resp. $\mathbb{P}^n$) is an affine (resp. projective) algebraic set.
d) Prove that any two distinct lines in $\mathbb{P}^2$ intersect at exactly one point.

4. a) Let $f(x_1, \ldots, x_n)$ be a degree d polynomial, and $Z = \{f = 0\} \subset \mathbb{A}^n$. Prove that any line in $\mathbb{A}^n$ is either completely contained in Z or interesects Z in at most d points.
b) Let $f(x_0, x_1, \ldots, x_n)$ be a degree d homogeneous polynomial, and $Z = \{f = 0\} \subset \mathbb{P}^n$. Prove that any line in $\mathbb{P}^n$ is either completely contained in Z or interesects Z in at most d points.

5. a) Consider the circle $Z = \{x^2 + y^2 = 1\}$ and the point $N = (0, 1)$. Let ℓ_m be the line of slope m through N. Prove that ℓ_m intersects Z at another point $P(m)$ and find the coordinates of $P(m)$.
b) Prove that m is rational if and only if $P(m)$ has rational coordinates.
c) Find all integer solutions of the equation $a^2 + b^2 = c^2$. *Hint: rewrite it as* $\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1$.