MAT 248A, Winter 2024
Homework 2

Due before 11:00 on Wednesday, January 24

Please write the homework solutions in connected sentences and explain your work. Mark the answers to each question. Scan or take pictures of your homework and upload it to Gradescope before due time.

In all problems we work over an arbitrary field $\mathbf{k}$ unless stated otherwise.

1. Let I and J be two ideals in $\mathbf{k}[x_1, \ldots, x_n]$.
a) Assume $f \in I \cap J$, prove that $f^2 \in I \cdot J$.
b) Use (a) to prove inclusions $I \cdot J \subset I \cap J \subset \sqrt{I \cdot J}$.
c) Prove $\sqrt{I \cdot J} = \sqrt{I \cap J}$.

2. Let $I = (y, y - x^2)$.
a) Find $Z(I)$.
b) Find $\sqrt{I}$.

3. Let $Z = \{a_0 x_0^d + a_1 x_1^d + \ldots + a_r x_r^d = 0\} \subset \mathbb{P}^n$ where $a_i \in \mathbf{k}$ are nonzero constants and $r \leq n$.
a) Describe the intersection of Z with affine charts U_i for all i.
b) Assume $\mathbf{k} = \mathbb{R}$. Find all singular points of Z.

4. a) Denote by $N(n, d)$ the dimension of the space of degree d homogeneous polynomials in n variables. Compute $N(n, d)$.
b) Suppose that $P_1, \ldots, P_k$ are points in $\mathbb{P}^n$ and $k < N(n+1, d)$. Prove that there exists a (nonzero) degree d homogeneous polynomial $f(x_0, \ldots, x_n)$ such that

$$f(P_1) = \ldots = f(P_k) = 0.$$

5. A **quadric** is a subset $\{f = 0\} \subset \mathbb{P}^n$ where f is a nonzero homogeneous polynomial of degree 2.
a) Given 5 points in $\mathbb{P}^2$, prove that there is a quadric containing all these points.
Hint: Use problem 4.
b) Given 9 points in $\mathbb{P}^3$, prove that there is a quadric containing all these points.
c) Given 3 lines in $\mathbb{P}^3$, prove that there is a quadric containing all these lines.
Hint: choose 3 points on each line and use (b).