MAT 248A, Winter 2024
Homework 3

Due before 11:00 on Wednesday, January 31

Please write the homework solutions in connected sentences and explain your work. Mark the answers to each question. Scan or take pictures of your homework and upload it to Gradescope before due time.

In all problems we assume that the ground field $\mathbf{k}$ is algebraically closed.

1. Suppose that $U \subset \mathbb{A}^n$ is a nonempty open subset and a polynomial $f(x_1, .., x_n)$ vanishes at all points of U. Prove that $f = 0$.

2. (Twisted cubic) Consider the map $\Phi : \mathbb{P}^1 \to \mathbb{P}^3$ defined in homogeneous coordinates by

$$[x_0 : x_1] \mapsto [x_0^3 : x_0^2 x_1 : x_0 x_1^2 : x_1^3]$$

a) Prove that Φ is well defined, that is, points with equivalent coordinates are mapped to points with equivalent coordinates. Also, prove that Φ is injective.
b) Prove that the point $[y_0 : y_1 : y_2 : y_3]$ belongs to the image of Φ if and only if

$$y_0 y_2 = y_1^2,\ y_1 y_3 = y_2^2,\ y_0 y_3 = y_1 y_2.$$

c) Prove that the image of Φ is an algebraic set in $\mathbb{P}^3$.
d) Let P, Q, R be three distinct points in $\mathbb{P}^1$. Prove that there is a unique 2-plane in $\mathbb{P}^3$ containing the points $\Phi(P), \Phi(Q), \Phi(R)$. *Hint: use Vandermonde determinant.*

3. (Segre embedding) Let $N = (m+1)(n+1) - 1$. Consider the map $S : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^N$ defined by

$$S([x_0 : \ldots : x_n], [y_0 : \ldots : y_m]) = [x_i y_j],\ 0 \le i \le n,\ 0 \le j \le m.$$

There are $N + 1 = (n+1)(m+1)$ homogeneous coordinates in the right hand side which can be thought of as entries in a $(n+1) \times (m+1)$ matrix.
a) Prove that S is well-defined and injective.
b) Prove that the matrix $(x_i y_j)$ has rank 1 and any rank 1 matrix appears this way.
c) Use (b) to prove that the image of S is a closed subset of $\mathbb{P}^N$.

4. Prove that for $n = m = 1$ the image of the Segre embedding $S : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$ is a quadric in $\mathbb{P}^3$.

5. Suppose that $Z_1 \subset \mathbb{P}^n$ and $Z_2 \subset \mathbb{P}^m$ are closed subsets. Prove that $S(Z_1 \times Z_2)$ is closed in $\mathbb{P}^N$.