MAT 248A, Winter 2024
Homework 4

Due before 11:00 on Wednesday, February 7

Please write the homework solutions in connected sentences and explain your work. Mark the answers to each question. Scan or take pictures of your homework and upload it to Gradescope before due time.

1. Let $\mathbf{k}$ be an arbitary field such that char $\mathbf{k} \neq 2$, V a vector space over $\mathbf{k}$, and $Q(u,v)$ a symmetric bilinear form on V.

a) The kernel K of the form Q is defined as the set of vectors v such that $Q(u,v) = 0$ for all $u \in V$. Prove that K is a vector subspace of V and Q induces a well defined symmetic bilinear form on V/K with no kernel.

b) Assume Q has no kernel, prove that there is a vector v such that $Q(v,v) \neq 0$.

c) Assume Q has no kernel, and $Q(v,v) \neq 0$. Define $v^{\perp} = \{u : Q(u,v) = 0\}$. Prove that $V = \langle v \rangle \bigoplus v^{\perp}$.

d) Use parts (a)-(c) to prove that for any Q there exists a basis $\{v_1, \ldots, v_n\}$ of V where the matrix $Q(v_i, v_j)$ is diagonal.

2. a) Let $F(x_0, \ldots, x_n)$ be a homogeneous polynomial of degree 2. Prove that there is a symmetric bilinear form Q on the $(n+1)$-dimensional space with basis $\{e_0, \ldots, e_n\}$ such that $F(x_0, \ldots, x_n) = Q(x_0e_0 + \ldots + x_ne_n, x_0e_0 + \ldots + x_ne_n)$.

b) Assume $\mathbf{k}$ is algebraically closed and char $\mathbf{k} \neq 2$. Use problem 1 to prove that the quadric $\{F(x_0, \ldots, x_n) = 0\} \subset \mathbb{P}^n$ is isomorphic to $\{x_0^2 + \ldots + x_r^2 = 0\}$ for some $r \leq n$.

c) In the notations of (b), prove that the quadric is smooth if and only if $r = n$.

3. Prove that any smooth quadric in $\mathbb{P}^2$ is isomorphic to $\mathbb{P}^1$.

4. Prove that any smooth quadric in $\mathbb{P}^3$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. *Hint: use Segre embedding.*

5. Prove that any smooth quadric in $\mathbb{P}^3$ contains infinitely many lines, and describe all such lines.