MAT 248A, Winter 2024
Homework 5

Due before 11:00 on Wednesday, February 14

Please write the homework solutions in connected sentences and explain your work. Mark the answers to each question. Scan or take pictures of your homework and upload it to Gradescope before due time.

1. Describe the preimages of the curves under the blowup of $\mathbb{C}^2$ at the origin:
a) $\{xy = 0\}$
b) $\{x^2 = y^3\}$
c) $\{x^3 = y^5\}$

2. Prove that the set of $m \times n$ matrices of maximal rank is a Zariski open subset of the space $\mathbb{A}^{mn}$ of all matrices. *Hint: a matrix has maximal rank if at least one of its maximal minors is nonzero*

3. Let $M_k(m \times n)$ be the set of $m \times n$ matrices of rank exactly k.
a) Describe the sets $M_k(2 \times 2)$ for all k by explicit equations (and inequalities).
b) Given $1 \le i_1 < \ldots < i_k \le m$, let $U_{i_1,\ldots,i_k}$ be the subset of $M_k(m \times n)$ such that the rows $i_1, \ldots, i_k$ are linearly independent. Use problem 2 to prove that $U_{i_1,\ldots,i_k}$ is isomorphic to a Zariski open subset in some affine space $\mathbb{A}^N$.
c) Use (b) to prove that M_k is a smooth manifold (with open charts $U_{i_1,\ldots,i_k}$) and find its dimension.

4. Let $M_{\le k}(m \times n)$ be the set of $m \times n$ matrices of rank at most k.
a) Describe the sets $M_{\le k}(2 \times 2)$ for all k by explicit equations.
b) Prove that $M_{\le k}(m \times n)$ is Zariski closed in the space $\mathbb{A}^{mn}$ of all matrices.
c) Prove that $M_k(m \times n)$ is Zariski open in $M_{\le k}(m \times n)$.

5. Let $X \subset \mathbb{A}^4 \times \mathbb{P}^1$ be the space of pairs (A, ℓ) where $A \in \mathbb{A}^4$ is a 2×2 matrix of rank at most 1, $\ell \in \mathbb{P}^1$ is a line in $\mathbb{A}^2$ through the origin, and $A\ell = 0$.
a) Prove that X is a Zariski closed subset of $\mathbb{A}^4 \times \mathbb{P}^1$.
b) Describe the fibers of the projection $X \to \mathbb{A}^4$.
c) Describe the fibers of the projection $X \to \mathbb{P}^1$.