MAT 248A, Winter 2024
Homework 6

Due before 11:00 on Wednesday, February 28

Please write the homework solutions in connected sentences and explain your work. Mark the answers to each question. Scan or take pictures of your homework and upload it to Gradescope before due time.

1. Let $X = \mathbb{A}^1 \setminus \{0\}$.
a) Describe all regular functions on X.
b) Describe all regular morphisms $X \to X$.
c) Describe all regular morphisms $(X \times X) \to (X \times X)$.

The **Grassmannian** $\mathrm{Gr}(k,n)$ is the space of all k-dimensional subspaces of the n-dimensional vector space. In the following problems we describe $\mathrm{Gr}(k,n)$ as an algebraic variety.

2. a) Prove that $\mathrm{Gr}(k,n)$ can be identified with the space of $k \times n$ matrices of rank k, modulo row operations.
b) Given $1 \leq i_1 < \ldots < i_k \leq n$, let $U_{i_1,\ldots,i_k}$ be the subset of $\mathrm{Gr}(k,n)$ such that the columns $i_1, \ldots, i_k$ are linearly independent. Prove that $U_{i_1,\ldots,i_k}$ is isomorphic to $\mathbb{A}^{k(n-k)}$. *Hint: use row operations to change the submatrix in columns $i_1, \ldots, i_k$ to identity matrix.*
c) Prove that $\mathrm{Gr}(k,n)$ is a smooth manifold of dimension $k(n-k)$.

3. Let M be a $k \times n$ matrix of rank k, define $\Delta_{i_1,\ldots,i_k}(M)$ to be the $k \times k$ minor in columns $i_1, \ldots, i_k$. These are called *Plücker coordinates.*
a) Suppose that M and M' are related by row operations. Prove that there exists a nonzero constant λ such that

$$\Delta_{i_1,\ldots,i_k}(M') = \lambda \Delta_{i_1,\ldots,i_k}(M).$$

b) Prove that $U_{i_1,\ldots,i_k} = \{M : \Delta_{i_1,\ldots,i_k}(M) \neq 0\}$ (the right hand side is well defined by part (a)), and write the coordinates on $U_{i_1,\ldots,i_k}$ in terms of Plücker coordinates.

4. Use problem 3 to prove that $\Delta_{i_1,\ldots,i_k}(M)$ for all possible k-element subsets define an injective map $\phi : \mathrm{Gr}(k,n) \to \mathbb{P}^N$ where $N = \binom{n}{k} - 1$.

5. a) Let M be a 2×4 matrix, prove that $\Delta_{1,2}\Delta_{3,4} + \Delta_{1,4}\Delta_{2,3} = \Delta_{1,3}\Delta_{2,4}$.
b) Prove that $\mathrm{Gr}(2,4)$ is isomorphic to a quadric in $\mathbb{P}^5$.