MAT 248A, Winter 2024
Homework 7

Due before 11:00 on Wednesday, March 6

Please write the homework solutions in connected sentences and explain your work. Mark the answers to each question. Scan or take pictures of your homework and upload it to Gradescope before due time.

1. In this problem we work over the field $\mathbb{R}$.
a) Draw the affine variety $X = \{x^2 + y^2 = 0\} \subset \mathbb{A}^2(\mathbb{R})$, prove that it is a smooth real manifold and find its dimension.
b) Find the dimension of X in the sense of algebraic geometry (defined by chains of irreducible closed subsets).
c) Find the dimension of Zariski tangent space at all points of X.

2. a) Draw the affine variety $X_{\mathbb{R}} = \{y^2 = x^3 - x^2\} \subset \mathbb{A}^2(\mathbb{R})$, and find all of its singular points.
b) Find all singular points of $X_{\mathbb{C}} = \{y^2 = x^3 - x^2\} \subset \mathbb{A}^2(\mathbb{C})$.

3. Suppose that $p(x)$ is a polynomial with distinct roots. Prove that $\{y^2 = p(x)\}$ is a smooth 1-dimensional variety (over $\mathbb{R}$ and over $\mathbb{C}$).

4. Let $X = \{f_1 \cdots f_s = 0\} \subset \mathbb{A}^n$ where the polynomials $f_i(x_1, \ldots, x_n)$ are irreducible and pairwise distinct.
a) Prove that X is singular at all points where at least two of f_i vanish.
b) Prove that all singular points of X are the ones in (a) and the singular points of the components $\{f_i = 0\}$.

5. Find all c such that the variety $\{x^2 + y^2 + z^2 = 1,\ x + 2y + 3z = c\} \subset \mathbb{A}^3$ is smooth. Here you can work over $\mathbb{R}$ or over $\mathbb{C}$.