MAT 248A, Winter 2024
Homework 8

Due before 11:00 on Friday, March 15

Please write the homework solutions in connected sentences and explain your work. Mark the answers to each question. Scan or take pictures of your homework and upload it to Gradescope before due time.

1. Let $X = \{\sum_{i,j=1}^n a_{ij}x_ix_j = 1\}$ be a quadric in $\mathbb{A}^n$ defined by a symmetric matrix $A = (a_{ij})$. Describe the Zariski tangent space of X at every point in terms of the matrix A.

2. Suppose that $X = \{f(x, y) = 0\}$ is a smooth curve in $\mathbb{A}^2$. Consider the differential form

$$\omega = \begin{cases} \frac{dx}{f_y} & \text{if } f_y \neq 0 \\ -\frac{dy}{f_x} & \text{if } f_x \neq 0 \end{cases}$$

where $f_x = \frac{\partial f}{\partial x}$ and $f_y = \frac{\partial f}{\partial y}$. Prove that ω is well-defined and regular everywhere on X.

3. a) Compute the form ω from problem 2 for $X = \{x^2 + y^2 = 1\}$.

b) Compute the integral $\int_{X(\mathbb{R})} \omega$.

4. In the notations of problem 2, consider the open subsets $U_1 = \{f_x \neq 0\} \cap X$ and $U_2 = \{f_y \neq 0\} \cap X$.

a) Prove that the module of differentials on U_1 is generated by dy, that is, any 1-form on U_1 can be written as $g(x, y)dy$ where $g(x, y)$ is a regular function on U_1. Similarly, prove that the module of differentials on U_2 is generated by dx.

b) Suppose that α is a regular 1-form on X. Prove that $\alpha = g(x, y)\omega$ where $g(x, y)$ is a regular function on X and ω is the form from problem 2. *Hint: consider the restrictions of α to U_1 and U_2 and use part (a).*

5. Let $S = \{z_0z_3 - z_1z_2 = 0\}$ be the Segre quadric in $\mathbb{P}^3$. Recall from HW 4 that S contains two families of lines. Let p be an arbitrary point on S.

a) Suppose that a line L is contained in S, prove that $T_pL \subset T_pS$.

b) Prove that $T_pS \cap S$ is a pair of lines. *Hint: prove that it is a quadric in T_pS and use part (a).*

c) Use (a) and (b) to prove that any line in S belongs to one of two families.

6*. (**This is a bonus problem**) Suppose L_1, L_2, L_3, L_4 are 4 lines in general position in $\mathbb{P}^3$. How many lines L intersect all four?