Algebraic Varieties

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Preface

This book is largely intended as a substitute for Chapter I (and an invitation to Chapter IV) of Hartshorne [**Har77**], to be taught as an introduction to varieties over a quarter- or semester-long course. The main substantive difference is that it incorporates the concept of abstract varieties defined using atlases, as is commonly done for differentiable manifolds. Although this is, as far as I am aware, ahistorical,[1] it seems to strike a good balance between keeping the material accessible while providing sufficient generality to serve as the foundation for, e.g., learning about toric varieties. This opens the door to treating abstract curves in a more geometric manner, and we also include material on complete varieties and (the geometric forms of) the valuative criteria in the absolute case. As a result, we give a very geometric proof of the fact that every global regular function on a projective variety is constant. We have also included material on differential forms, divisors on curves, and the analytic topology for complex varieties. Scattered through the book are some theorems with nontrivial proofs (for instance, Theorem 8.4.10 stating that any two points on a variety may be connected by a chain of curves), as well as some results whose proofs, while relatively short, illustrate important techniques in algebraic geometry, such as reduction to the universal case (Corollary 8.4.2) and the study of varieties of interest (such as the secant variety in 9.5.5) by introducing an auxiliary variety in which existence conditions are replaced by choices of objects.

Following [**Har77**], we have taken a regular-function-centered approach to defining morphisms of varieties, which thus leads naturally into scheme theory. Indeed, due to its origins in courses taught from [**Har77**], much of our presentation follows [**Har77**] quite closely. For material not found in [**Har77**], many of the proofs follow Shafarevich [**Sha94a**], [**Sha94b**], with some influence also from Mumford [**Mum99**].

Obviously, the book is still in early draft form. I have not yet included many exercises, with the most glaring omissions being example-oriented exercises, and exercises in early chapters. Also, some topics like complete local rings and blow-ups are not yet included.

Conventions. All rings (including algebras over rings) are assumed to be commutative, with multiplicative identity. In particular, an R-algebra is equivalent to a ring S, together with a homomorphism $R \to S$.

Compact topological spaces are not assumed Hausdorff.

[1] As I understand it, abstract varieties were first introduced by Weil over arbitrary fields, requiring much more technical definitions.

Contents

CHAPTER 1

Introduction: an overview of algebraic geometry through the lens of plane curves

Algebraic geometry is a technical subject, and of necessity, much of this book is devoted to definitions and results of a rather general and abstract nature. We therefore begin much more concretely, by introducing a number of important concepts in a very down-to-earth context. To that end, we will discuss plane curves, and elliptic curves in particular. Many of the concepts alluded to here will be returned to (with varying degrees of completeness) later in the book, while a few serve only as hints of what lies beyond the scope of what we will cover.

1.1. Plane curves

Algebraic geometry is, in brief, the study of sets of solutions of (possibly multiple) polynomials in multiple variables. It is the interplay between the geometry of the solution set and the algebra of the polynomials that gives algebraic geometry its richness. In algebraic geometry, few objects are as basic as the plane curve. We start with a single polynomial equation $f(x, y) = 0$ in two variables, which we will assume for the sake of simplicity has coefficients in $\mathbb{Z}$, so that we can conveniently consider the curve as defined over $\mathbb{Q}$ or over $\mathbb{Z}/p\mathbb{Z}$ for any p, or in fact over any field. There are then immediately a range of questions one can ask about the plane curve defined as the set of solutions of this equation:

- What do the real points of the curve look like? The complex points?
- Does the curve have a point with coordinates in $\mathbb{Q}$?
- What can one say about the number of solutions over $\mathbb{Q}$? Over a number field? Over a finite field?

Generally, if $f(x, y)$ is linear or quadratic, these questions can be answered in a rather complete manner. However, the cubic case becomes much deeper, and is the subject of the theory of elliptic curves. Before delving into this, we investigate some more general ideas.

We now begin discussing some basic properties of the geometry and topology of plane curves.

Smoothness. Over $\mathbb{R}$ or $\mathbb{C}$, it makes sense to ask whether the curve defined by $f(x, y) = 0$ is a (one-dimensional) manifold. By the implicit function theorem, one can check that this is the case if there is no point with

$$f(x_0, y_0) = \frac{\partial f}{\partial x}(x_0, y_0) = \frac{\partial f}{\partial y}(x_0, y_0) = 0.$$

Over $\mathbb{C}$, the converse also holds. For a plane curve defined by $f(x, y) = 0$, if this property is satisfied we say it is **smooth** or **nonsingular**. Otherwise, any point satisfying the above simultaneous vanishings is called a **singularity** of the curve.

Warning 1.1.1. Over non-algebraically-closed fields, the notions of smooth and non-singular are not equivalent, but they are closely related, and for the purposes of this introduction we will use them interchangeably.

These ideas will be explored in more detail in Chapter 4.

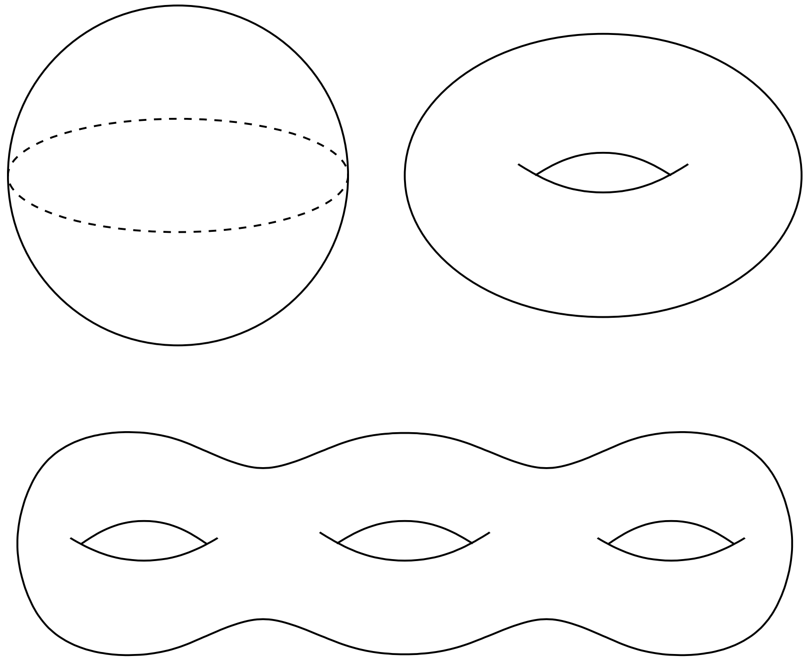

FIGURE 1. Complex plane curves of genus 0, 1 and 3.

Compactness. Another property we might wish to study is compactness. If we consider plane curves over $\mathbb{R}$, we see that many of them (e.g., a line, or $xy = 1$, or $y = x^2$, or $y^2 = x^3 - x$) have points going "to infinity", and therefore are not compact (in the usual topology on $\mathbb{R}^2$). There are some exceptions, like the circle. However, over $\mathbb{C}$ it turns out that a plane curve *always* has points going to infinity, and is therefore *never* compact.

We wish to compactify the picture by adding "points at infinity," and the way to accomplish this is to work not in the affine plane $\mathbb{A}^2$, but in the **projective plane** $\mathbb{P}^2$.

Informally, one may think of the projective plane as the affine plane together with one "point at infinity" for each line through the origin in the affine plane. A plane curve goes through a given "point at infinity" if it has a branch with slope approaching the slope of the corresponding line.

More formally, we can think of the projective plane as having three homogeneous coordinates X, Y, Z, with the points of $\mathbb{P}^2$ corresponding to triples (X_0, Y_0, Z_0) with not all three values equal to 0, and considered up to simultaneous scaling. With this description, the affine plane could be the open subset with $Z_0 \neq 0$; on this subset, we can always scale so that $Z_0 = 1$, so we find that the points correspond to points $(x, y, 1)$, as they should. The points at infinity are then the points with $Z_0 = 0$.

In this context, we observe that although the value of a polynomial $F(X, Y, Z)$ is no longer well-defined, if F is **homogeneous** (that is, has all monomials of equal total degree) then whether or not it is zero at (X_0, Y_0, Z_0) remains well defined. Thus, we can talk about the zero set of a homogeneous polynomial in $\mathbb{P}^2$. If we have started with $f(x, y) = 0$ in the (affine) plane, we can obtain the closure in projective space by setting the associated polynomial $F(X, Y, Z)$ to be the same as f, with X and Y in place of x and y, and adding in just enough powers of Z to each term so that the total degree becomes constant.

These ideas are explored in Chapters 6 and 8.

In order to talk about smoothness of a projective curve defined by $F(X,Y,Z)=0$, we can simply apply our definition of smoothness for affine curves to the three curves $F(x,y,1)=0$, $F(x,1,z)=0$, and $F(1,y,z)=0$.

A few theorems. Having discussed smoothness and projective space, we can now address one of the above questions with the following:

Theorem 1.1.2. *Let C be a smooth curve in $\mathbb{P}^2$ over $\mathbb{C}$. Then the (complex) points of C naturally form a compact, connected, orientable surface (in the sense of a 2-dimensional real manifold).*

This combines Corollary A.2.4, Corollary A.4.1, and Theorem A.5.1.

Using classification of surfaces, we can then define:

Definition 1.1.3. A complex smooth projective plane curve has **genus** g if the associated topological surface has genus g; that is, if it has g "holes".

We can now state another theorem, the "degree-genus formula," which begins to hint at the interplay between algebra and geometry which arises in the subject.

Theorem 1.1.4. *Let C be a smooth projective plane curve defined as the zero set of a polynomial $F(X,Y,Z)$. If F has degree d, and C has genus g, then*

$$g=\frac{(d-1)(d-2)}{2}.$$

See Exercise 11.2.6 for a proof of this. Note that not every genus can occur in this way (for instance, genus 2 is not possible). It turns out that every genus can occur for smooth projective curves imbedded in projective three-space, and in fact, every smooth projective curve can be imbedded in projective three-space. In fact, we will introduce a more general notion of abstract curve, and see in Chapter 7 that every abstract smooth curve can be realized as an open subset of a smooth projective curve, so in fact every smooth curve can be imbedded in projective three-space.

1.2. Elliptic curves

We now specialize from plane curves to elliptic curves. For our purposes, we start with the following simple definition:

Definition 1.2.1. The curve defined by $f(x,y)=0$ is an **elliptic curve** if $f(x,y)$ is of the form y^2-x^3-ax-b for some a,b, and $\Delta:=-16(4a^3+27b^2)$ is non-zero. We then say that $f(x,y)$ is in **Weierstrass form**.

Why this definition? Over a field of characteristic $\neq 2,3$, any plane curve given by an irreducible cubic polynomial can always be put in the above form after appropriate (linear) change of variables. The condition that $\Delta\neq 0$ is more interesting, and already provides an indication of the interplay between algebra and geometry which is so characteristic of algebraic geometry.

Proposition 1.2.2. *For $f(x,y)$ in Weierstrass form, smoothness is equivalent to x^3+ax+b having distinct roots, which is equivalent to having $\Delta\neq 0$.*

Following our previous discussion, we will frequently wish to consider the corresponding projective curve, given by the homogeneous equation $F(X,Y,Z)=Y^2Z-X^3-aXZ^2-bZ^3=0$.

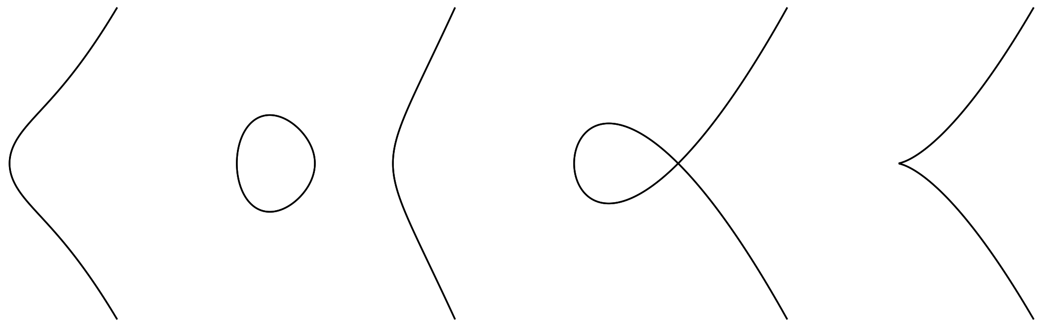

FIGURE 2. Four possibilities for real cubic curves in Weierstrass form.

Real points. We first consider the real points of the curve. Here, it is not hard to see that the only possibilities for a curve in Weierstrass form are: a connected smooth curves with branches going off to infinity; a smooth curve with one component homeomorphic to a circle, and the other having branches going off to infinity; a curve with a "node", i.e., a point at which it crosses itself;[1] or a curve with a "cusp", where it has a sharp point. Because of the hypothesis that $\Delta \neq 0$, only the first two correspond to elliptic curves.

We next note that the unbounded branches both have slopes which approach vertical, so the unique point at infinity which we have to add is the one which corresponds to a vertical line. We can see this explicitly from the above projective equation: points at infinity correspond to $Z = 0$, and in this case we must have $X^3 = 0$, and hence $X = 0$, so the point at infinity (in projective coordinates) must be $(0, 1, 0)$.

In general, when one projectivizes in this manner, one may inadvertently introduce singular points at infinity. However, one checks easily that the point at infinity of an elliptic curve is always a smooth point, so we see that under our hypothesis that $\Delta \neq 0$, we have that the corresponding curve is smooth not only in the affine plane, but also in the projective plane.

Complex points. As we have already mentioned, the complex points of an elliptic curve will form an orientable surface, and we see from the degree-genus formula that it will have genus 1. We thus conclude that it is topologically the same as $\mathbb{C}/\Lambda$, where Λ is a lattice. In fact, this holds in a stronger sense:

THEOREM 1.2.3. *Every elliptic curve E has genus 1. In fact, as a complex manifold, E is isomorphic to $\mathbb{C}/\Lambda$ for some lattice Λ in the complex plane.*

Thus, there is a map $\mathbb{C} \to \mathbb{C}/\Lambda \to E$, which is defined in terms of the Weierstrass $\wp$ function. However, while this map is complex analytic, it is very far from being algebraic – in fact, there is no nonconstant algebraic map from $\mathbb{C}$ to a complex elliptic curve.

We notice that the set $\mathbb{C}/\Lambda$ naturally has the structure of an abelian group, inherited from the addition law on $\mathbb{C}$. This means that for any elliptic curves considered over $\mathbb{C}$, the points have an abelian group structure as well.

Given this version of the group law, it is quite easy to describe the ***n*-torsion points** of E, i.e., the points P such that $nP = 0$. If Λ is generated by τ_1, τ_2, then the n-torsion points are of the form $\frac{a}{n}\tau_1 + \frac{b}{n}\tau_2$, for $0 \leqslant a, b \leqslant n$, so we see that the subgroup of n-torsion points is isomorphic to $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$.

[1]It is possible for the branches at which the curve crosses itself to be strictly imaginary, so that the node creates an isolated real point, as in the case $y^2 = x^3 - 3x - 2$.

Intersection theory and the group law. It is a remarkable fact that under the isomorphism $\mathbb{C}/\Lambda \xrightarrow{\sim} E$ of complex manifolds, the addition law on $\mathbb{C}/\Lambda$ gives rises to an *algebraically-defined* addition law on E. This law may be described intrinsically in terms of algebraic geometry using rudimentary **intersection theory**. This is one of many examples of an application of a problem in **enumerative geometry** (in this case, counting the number of points in the intersection of an elliptic curve and a line) to a problem having nothing to do with enumeration (constructing a group law on the set of points of an elliptic curve).

Suppose we have a line L in the projective plane. Our heuristic claim is as follows:

L intersects E in 3 points.

It is in fact true that L intersects E in at most 3 points. To make a precise statement that they intersect in exactly 3 points, we need to address three issues:

- L might miss E if we look at points over a non-algebraically closed field (e.g., $\mathbb{R}$);
- L intersects E in too few points if L happens to be tangent to E.

Notice that if we were working in the affine plane instead of the projective plane, there would be the additional issue that L intersects E in too few points if L happens to be vertical.

The first point is easily addressed: we work over an algebraically closed field, such as $\mathbb{C}$ (in fact, we will be able to give a statement in the end which does not require this restriction, but the situation is quite special to elliptic curves). However, the second point is more substantive. In this case, it can be addressed on ad-hoc basis by saying if L is tangent to E at a point P, the intersection point should count for 3 or 2 points, depending on whether P is or is not an inflection points of E. More generally, for plane curves one can make a complicated set of geometric axioms for **intersection multiplicity**; see for instance [**Kir92**, Thm. 3.18]. However, as one sees in an introductory schemes course, one of the great benefits of scheme theory (and particularly **non-reduced** schemes) is a very clean and simple definition of intersection multiplicity. In any case, once one has an appropriate definition of intersection multiplicity, our statement for lines and elliptic curves is a special case of Bezout's theorem:

THEOREM 1.2.4. *Let C, D be distinct smooth curves in the projective plane, of degrees d, e, and defined over an algebraically closed field k. Then the number of points of $C \cap D$, counting multiplicity, is precisely $d \cdot e$.*

In fact, the condition that C and D are smooth is not necessary. We give a proof of the theorem in the case that at least one is smooth in Exercise 9.3.20. Given this theorem, we can define an operation

$$* : E \times E \to E$$

by setting $P * Q$ to be the third point of intersection of E with the line L through P and Q (where if $P = Q$, we let L be the tangent line to E at that point).

If we denote by O the point at infinity of E, we can define the addition law

$$+ : E \times E \to E$$

by $P + Q := (P * Q) * O$. One then has:

THEOREM 1.2.5. *The operation $+$ defines an abelian group law on E, with the identity element given by O, and the inverse of a point (x, y) given by $(x, -y)$.*

In fact, everything is easy to check except the associativity of the operation. Elementary proofs of associativity can be given either algebraically or geometrically, but either approach is rather Byzantine. There are however very elegant proofs which take a more sophisticated approach, using Picard varieties.

Furthermore, if points P, Q have coefficients in some field k, it is easy to check that $P * Q$, and hence $P + Q$, will have coefficients in k:

Lemma 1.2.6. *The group law on E makes $E(k)$, the points of E with coefficients in k, into a group, for any field k in which $\Delta(E) \neq 0$.*

One can also show:

Theorem 1.2.7. *The group law on $E(\mathbb{C})$ defined by intersection theory is the same as the group law on $E(\mathbb{C})$ defined by an isomorphism $\mathbb{C}/\Lambda \xrightarrow{\sim} E(\mathbb{C})$ for some lattice Λ.*

From here, it is not hard to prove:

Corollary 1.2.8. *If k is an algebraically closed field of characteristic 0, then the subgroup of n-torsion points of $E(k)$ is isomorphic to $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$.*

This is a basic example of the usefulness of studying an algebraic variety from a complex-analytic perspective.

The j-invariant. It turns out that while Weierstrass equations are a useful tool, they are not ideal for classifying elliptic curves, because two different Weierstrass equations may give the "same" curve (for instance, differing only by a change of coordinates). For this, one uses instead the following:

Definition 1.2.9. The **j-invariant** of an elliptic curve $y^2 = x^3 + ax + b$ is given by

$$j(E) := 1728 \frac{4a^3}{4a^3 + 27b^2}.$$

One can define in general what it should mean for curves to be **isomorphic** as abstract algebraic curves, and then it is not too hard to prove the following:

Proposition 1.2.10. *Let k be an algebraically closed field. Two elliptic curves over k are isomorphic (over k) if and only if they have the same j-invariant.*

Thus, this can be considered to give a classification of elliptic curves: they are in correspondence with elements of k. Notice that we can then think of a "moduli space" of elliptic curves, meaning a space whose points correspond to elliptic curves. In this case, the space is simply the affine line $\mathbb{A}^1$. This concept of moduli space can be generalized to curves of higher genus as well, and the geometry of moduli spaces of curves gets much more complicated as the genus increases. Indeed, moduli spaces of curves remain an active subject of research today.

CHAPTER 2

Affine algebraic varieties

In this chapter, we study algebraic sets and algebraic varieties in affine space, relating them to ideals in polynomial rings, and developing the basic properties of the Zariski topology.

2.1. Zero sets and the Zariski topology

The basic idea is that an affine algebraic set should be the zero set of a system of polynomial equations in several variables.

EXAMPLE 2.1.1. The zero set of the polynomial 1 is the empty set.

The problem with this point of view is that over $\mathbb{R}$ (or $\mathbb{Q}$), the zero set of $f(x,y) = x^2 + y^2 + 1$ is also empty. But we don't want to consider it to be the "same variety," since it is nonempty over $\mathbb{C}$ (or $\mathbb{Q}(i)$).

Our solution, for the purposes of the present book, is that unless we explicitly say otherwise, we will always assume we are working with a field k which is algebraically closed.

With this hypothesis, we make some definitions:

DEFINITION 2.1.2. Affine space of dimension n over k, denoted by $\mathbb{A}^n_k$ (or $\mathbb{A}^n$ when k is prespecified) is the set of n-tuples of elements of k. An element $P \in \mathbb{A}^n_k$ is a **point**, and if $P = (a_1, \dots, a_n)$, the a_i are the **coordinates** of P.

We will use the following running notation:

NOTATION 2.1.3. Let $A_n = k[x_1, \dots, x_n]$ be the polynomial ring in n variables over k.

Then any $f \in A_n$ gives a function $\mathbb{A}^n_k \to k$, defined by

$$P = (a_1, \dots, a_n) \mapsto f(a_1, \dots, a_n).$$

Thus, the zeroes of f are a subset of $\mathbb{A}^n_k$.

DEFINITION 2.1.4. Given $T \subseteq A_n$, the **zero set** of T is defined by

$$Z(T) = \{P \in \mathbb{A}^n_k : f(P) = 0 \, \forall f \in T\}.$$

$X \subseteq \mathbb{A}^n_k$ is an **algebraic set** if there exists $T \subseteq A_n$ such that $X = Z(T)$.

EXAMPLE 2.1.5. With the algebraically closed assumption, if we consider the zero set in $\mathbb{A}^2$ of $f(x,y) = x^2 + y^2 + 1$, we can describe it quite explicitly. For each value of x, we can set $y = \sqrt{-x^2 - 1}$, and we get one or two points. We get two points if $x^2 + 1 \neq 0$ and $\operatorname{char} k \neq 2$, and we get one point otherwise.

The most basic properties of algebraic sets are expressed by the following proposition, which says (in an explicit way) that algebraic sets satisfy the conditions to form the closed sets of a topological space.

PROPOSITION 2.1.6. *We have the following:*

(1) $\bigcup_{i=1}^m Z(T_i) = Z(\prod_{i=1}^m T_i)$ *for any* $T_1, \dots, T_m \subseteq A_n$.

(2) $\bigcap_{i\in I} Z(T_i) = Z(\bigcup_{i\in I} T_i)$ *for any (possibly infinite) set* $\{T_i \subseteq A_n : i \in I\}$.
(3) $Z(1) = \emptyset$, $Z(0) = \mathbb{A}^n_k$.

Proof. (1) Induct on m. The case $m = 1$ is trivial. For $m > 1$, by the induction hypothesis we have $\bigcup_{i=1}^{m-1} Z(T_i) = Z(\prod_{i=1}^{m-1} T_i)$, so it is enough to treat the case $m = 2$. Now, if $P \in Z(T_1) \bigcup Z(T_2)$, then $P \in Z(T_i)$ for $i = 1$ or 2, which means either $f(P) = 0$ for all $f \in T_1$ or $f(P) = 0$ for all $f \in T_2$. Either way, we conclude that $f(P) = 0$ for any $f \in T_1T_2$, so $P \in Z(\prod_{i=1}^{m-1} T_i)$. On the other hand, suppose that $P \in Z(T_1T_2)$, and suppose that $P \notin Z(T_1)$. Then for some $f \in T_1$, we have $f(P) \neq 0$. But $fg(P) = 0$ for all $g \in T_2$ by hypothesis, so we conclude that $g(P) = 0$ for all $g \in T_2$, and hence $P \in Z(T_2)$.

(2) and (3) are clear from the definitions. □

The proposition allows us to define a topology on the set $\mathbb{A}^n_k$.

Definition 2.1.7. The **Zariski topology** on $\mathbb{A}^n_k$ is the topology whose closed sets are the algebraic sets.

We will be thinking of algebraic sets as topological spaces, using the subspace topology. However, ultimately this will not be enough to capture their structure, leading us also to consider what functions we should be studying on them.

Remark 2.1.8. The Zariski topology is compatible with classical topologies (e.g., when $k = \mathbb{R}$ or $\mathbb{C}$) in the sense that Zariski closed subsets are also closed in classical topologies. But the Zariski topology is much coarser, and is consequently a very pathological topology! For instance, it is almost never Hausdorff, but it is always compact (if one does not require Hausdorff in the definition of compactness). A substantial portion of the foundations of algebraic geometry involve dealing with the deficiencies of this topology.

Definition 2.1.9. A topological space X is **irreducible** if for all $X_1, X_2 \subseteq X$ closed with $X_1 \cup X_2 = X$, we have $X_1 = X$ or $X_2 = X$.

Remark 2.1.10. Irreducibility is equivalent to every open subset being dense, so an irreducible topological space has a *very* coarse topology. The only irreducible Hausdorff space is a point!

Definition 2.1.11. An **affine (algebraic) variety** is an irreducible closed subset of $\mathbb{A}^n_k$, in the Zariski topology (i.e., an irreducible algebraic set).

Example 2.1.12. $Z(xy) \subseteq \mathbb{A}^2$ is not irreducible, since it is equal to $Z(x) \cup Z(y)$.

A point is obviously irreducible. A slightly less trivial example:

Example 2.1.13. The Zariski topology on $\mathbb{A}^1$: we claim the closed sets are the finite sets (including the empty set), together with all of $\mathbb{A}^1$. Indeed, given any finite set of elements of k, it can be expressed as $Z(f)$ for some $f \in k[x]$. On the other hand, given a nonempty $T \subseteq k[x]$, if $T = (0)$ then $Z(T) = \mathbb{A}^1$. Otherwise, T contains some $f \neq 0$, and $Z(T) \subseteq Z(f)$, which is a finite set. This proves the claim.

Note that in particular, we conclude that because algebraically closed fields are always infinite, $\mathbb{A}^1$ is irreducible, hence a variety.

In higher dimensions, it is harder to use such direct analysis to prove irreducibility, and the connection to algebra will be very helpful.

We will show in Proposition 2.3.6 below that every algebraic set can be expressed as a finite union of algebraic varieties. This is why it is common to restrict attention primarily to the irreducible case.

Warning 2.1.14. The definition of "variety" is not completely standard: sometimes authors do not require irreducibility. The ambiguity is even more pronounced over non-algebraically-closed fields.

Exercise 2.1.15. Identifying the sets $\mathbb{A}_k^2 = k^2 = \mathbb{A}_k^1 \times \mathbb{A}_k^1$, describe the closed subsets in the product topology, and show that the Zariski topology on $\mathbb{A}_k^2$ is strictly finer than the product topology on $\mathbb{A}_k^1 \times \mathbb{A}_k^1$.

Exercise 2.1.16. Let $S \subseteq \mathbb{A}_k^n$ be a subset with the property that if $(a_1, \dots, a_n) \in S$, then $(\lambda a_1, \dots, \lambda a_n) \in S$ for all $\lambda \in k$. Suppose $f \in I(S)$, and write $f = f_0 + f_1 + \cdots + f_d$, where each f_i is homogeneous of degree i. Then show that $f_i \in I(S)$ for each i.

Exercise 2.1.17. Let X be a topological space.

(a) Show that if $Y \subseteq X$ is dense and irreducible, then X is also irreducible.
(b) Show that if X is the continuous image of an irreducible space, then X is irreducible.
(c) Suppose $Y \subseteq X$ has the following property: for every $P \in X \smallsetminus Y$, there exists a space Z and a continuous map $f : Z \to X$ such that $P \in f(Z)$ and $f^{-1}(Y)$ is dense in Z. Then show that Y is dense in X.

2.2. Zero sets and ideals

In order to relate algebraic sets to algebra, the fundamental observation is the following: given $T \subseteq A_n$, if I is the ideal generated by T, then we have $Z(T) = Z(I)$. So when dealing with algebraic sets and the Zariski topology, it is enough to consider sets $Z(I)$, where $I \subseteq A_n$ is an ideal.

This is very helpful, since the set of ideals of A_n has a lot more structure than the set of subsets of A_n!

As a first illustration of the usefulness of the algebraic point of view, one might wonder whether there are some algebraic sets which require infinitely many polynomials to define. This is ruled out by the following basic theorem of commutative algebra:

Theorem 2.2.1 (Hilbert basis). *If R is a polynomial ring in finitely many variables over a Noetherian ring, then R is Noetherian.*

Recall that a ring has every ideal finitely generated if and only if every ascending chain of ideals stabilizes (§1.4 of [**Eis95**]). Such a ring is **Noetherian**. For the proof of the theorem, see Theorem 1.2 of [**Eis95**].

As an immediate consequence since a field is Noetherian, we conclude:

Corollary 2.2.2. *Every algebraic set in $\mathbb{A}^n$ is $Z(f_1, \dots, f_m)$ for some finite set $f_1, \dots, f_m \in A_n$.*

In order to more fully explore the relationship between algebraic sets and ideals, the next question to address is:

Question 2.2.3. When do we have $Z(I) = Z(J)$ for two different ideals I, J of A_n?

Example 2.2.4. In $\mathbb{A}^1$, we have $Z(f) = Z(f^2)$ for any $f \in k[x]$, since f and f^2 have the same set of zeroes. More generally, if we have $f, g \in k[x]$ such that for m sufficiently large, $f | g^m$ and $g | f^m$, then $Z(f) = Z(g)$, and in fact the converse also holds. This turns out to generalize.

Recall that if $I \subseteq A_n$ is an ideal, then

$$\operatorname{rad} I := \{f \in A_n : f^m \in I \text{ for some } m \in \mathbb{N}\}.$$

Then because taking powers doesn't change the zero set of a polynomial, it is clear that:

Proposition 2.2.5. *Given $I \subseteq A_n$, we have $Z(I) = Z(\operatorname{rad} I)$.*

Note that this implies that for studying algebraic sets, it is enough to consider sets $Z(I)$ where I is a radical ideal (i.e., equal to its own radical).

In order to pursue this further, we define:

Definition 2.2.6. Given $Y \subseteq \mathbb{A}^n$, set $I(Y) = \{f \in A_n : f(P) = 0 \,\forall P \in Y\}$.

Observe that in symbols, to say $f \in I(Y)$ is equivalent to saying that $Y \subseteq Z(f)$.

It is clear that $I(Y)$ is always an ideal of A_n, and in fact, we easily see:

Proposition 2.2.7. *For any $Y \subseteq \mathbb{A}^n$, the set $I(Y)$ is a radical ideal.*

We also have the following basic observations:

Proposition 2.2.8. *Given $T_1, T_2 \subseteq A_n$ (not necessarily ideals), and $S_1, S_2 \subseteq \mathbb{A}^n$ (not necessarily algebraic sets), we have:*

(1) $T_1 \subseteq T_2 \Rightarrow Z(T_1) \supseteq Z(T_2)$.
(2) $S_1 \subseteq S_2 \Rightarrow I(S_1) \supseteq I(S_2)$.
(3) $I(S_1 \cup S_2) = I(S_1) \cap I(S_2)$.

We can then conclude the following:

Corollary 2.2.9. *If $S \subseteq \mathbb{A}^n_k$ is any subset, then $Z(I(S)) = \overline{S}$, the closure of S in the Zariski topology.*

If S is closed and irreducible, then $I(S)$ is a prime ideal.

Proof. For the first assertion, $Z(I(S))$ is closed by definition, and clearly contains S. On the other hand, given S' closed and containing S, we want to show that $S' \supseteq Z(I(S))$. Since $S' \supseteq S$, using (1) and (2) of Proposition 2.2.8 we find that $Z(I(S')) \supseteq Z(I(S))$. Since S' is assumed close, write $S' = Z(I')$ for some ideal I'; then $I(Z(I')) \supseteq I'$ by definition, so again using (1) of Proposition 2.2.8 we find

$$S' = Z(I') \supseteq Z(I(Z(I'))) = Z(I(S')) \supseteq Z(I(S)),$$

as desired.

Next, suppose that S is closed and irreducible. Suppose $fg \in I(S)$: then $S \subseteq Z(fg) = Z(f) \cup Z(g)$, so by irreducibility we conclude that $S \subseteq Z(f)$ or $S \subseteq Z(g)$, but this means precisely that $f \in I(S)$ or $g \in I(S)$. Hence, $I(S)$ is prime, as desired. ☐

We now use the following key result from commutative algebra. Following the tradition in commutative algebra, the name is left in German to make it more intimidating.

Theorem 2.2.10 (Hilbert Nullstellensatz[1]). *Given $I \subseteq A_n$ an ideal, and $f \in A_n$ such that $f(P) = 0$ for all $P \in Z(I)$, then $f^m \in I$ for some $m > 0$.*

See Theorem 1.6 of [**Eis95**]. Note that unlike the Hilbert basis theorem, the Nullstellensatz is special to polynomial rings over an algebraically closed field (although the difficult part of the proof can be formulated more generally; see Remark 2.2.22 below).

Corollary 2.2.11. *For any ideal $I \subseteq A_n$, we have $I(Z(I)) = \operatorname{rad}(I)$.*

If I is prime, then $Z(I)$ is irreducible.

[1] "Zero set lemma"

Proof. Indeed, the first part is just a rephrasing of the Nullstellensatz, noting that $I(Z(I)) = I(Z(\text{rad}(I)) \supseteq \text{rad}(I)$ is clear from the definitions.

For the second part, if I is prime, suppose $Z(I) = Y_1 \cup Y_2$ for $Y_i \subseteq \mathbb{A}^n_k$ closed. Then because prime ideals are radical, we have by the first part that $I = I(Z(I)) = I(Y_1) \cap I(Y_2)$. Then $I = I(Y_1)$ or $I = I(Y_2)$ by Exercise 2.2.12 below. We then have $Z(I) = Z(I(Y_1)) = Y_1$ or $Z(I) = Z(I(Y_2)) = Y_2$, proving the desired irreducibility. □

Exercise 2.2.12. Show that if $\mathfrak{p} \subseteq R$ is a prime ideal of a ring, and $\mathfrak{p} = I \cap J$ for ideals I, J, then $\mathfrak{p} = I$ or $\mathfrak{p} = J$.

We thus conclude the following key foundational result.

Theorem 2.2.13. *There is a one-to-one, inclusion-reversing correspondence*

$$\{\textit{algebraic sets } Y \subseteq \mathbb{A}^n_k\} \quad \longleftrightarrow \quad \{\textit{radical ideals } I \subseteq A_n\}$$

given by

$$\begin{array}{rcl} Y & \longmapsto & I(Y) \\ Z(I) & \longleftarrow & I. \end{array}$$

Moreover, under this correspondence, varieties correspond to prime ideals.

Example 2.2.14. Note that any point $(a_1, \dots, a_n)$ is always a closed subset, corresponding to the (maximal) ideal $(x_1 - a_1, \dots, x_n - a_n)$ (why is this maximal? Because if we mod out by it, we get k).

On the other hand, if $\mathfrak{m} \subseteq A_n$ is a maximal ideal, it corresponds to a minimal closed subset of $\mathbb{A}^n$, which must be a single point.

Thus, points of $\mathbb{A}^n_k$ correspond to maximal ideals of A_n. Moreover, we note that the ideal $(x_1 - a_1, \dots, x_n - a_n)$ is precisely the ideal of polynomials vanishing at $(a_1, \dots, a_n)$. Indeed, the latter ideal contains the former, so they must agree since we already know that $(x_1 - a_1, \dots, x_n - a_n)$ is maximal.

Now we can give many examples of varieties, starting with the most basic ones.

Example 2.2.15. $\mathbb{A}^n$ is a variety, since it corresponds to the prime ideal (0) of A_n.

Example 2.2.16. If $f \in A_n$ is an irreducible polynomial, then $Z(f)$ is a variety in $\mathbb{A}^n_k$, because A_n is a unique factorization domain, so f generates a prime ideal.

Definition 2.2.17. A variety of the form $Z(f)$ for $f \in A_n$ irreducible is a **hypersurface**.

We will see in Corollary 2.4.18 below that a hypersurface can be defined equivalently as a subvariety of $\mathbb{A}^n_k$ of codimension 1.

The following definition establishes the connection between algebraic sets and rings.

Definition 2.2.18. If $Y \subseteq \mathbb{A}^n_k$ is an algebraic set, the **affine coordinate ring** $A(Y)$ of Y is $A_n/I(Y)$.

The intuition is that this is the ring of algebraic functions on Y. If we have a polynomial, it gives a function of Y, but two polynomials give the same function if (in fact, if and only if) their difference is in $I(Y)$. The only part which is subtle is that it's not *a priori* clear that every algebraic function should come from a polynomial on all of $\mathbb{A}^n_k$, but we'll return to this later to justify it more fully.

Remark 2.2.19. Since $Z(I)$ is radical, we see that a coordinate ring is always nilpotent-free, and is finitely generated over k. Conversely, if a k-algebra R is nilpotent-free and generated over

k by $y_1, \dots, y_n$, then there is a surjection $k[x_1, \dots, x_n] \to R$ defined by $x_i \mapsto y_i$, and if the kernel of this surjection is denoted by I, we see that $R \cong A(Z(I))$. Thus, the possible coordinate rings of algebraic sets are precisely the nilpotent-free, finitely generated k-algebras.

In addition, we see that the coordinate rings of affine varieties are precisely the integral domains which are finitely generated k-algebras.

Exercise 2.2.20. Show that if $f(x, y) \in k[x, y]$ is of the form $y^2 - g(x)$ for a nonconstant polynomial $g(x) \in k[x]$, then $f(x, y)$ is irreducible, so that $Z(f(x, y))$ is irreducible and $I(Z(f(x, y))$ is generated by $f(x, y)$.

Exercise 2.2.21. All of our definitions so far make perfect sense even if k is not algebraically closed, and most of our basic results still hold. In fact, the only algebra theorem we have used which requires algebraic closure is the Nullstellensatz.

(a) Over your favorite non-algebraically closed field, give a counterexample to the Nullstellensatz.
(b) Show by example that if k is not algebraically closed, a prime ideal may have a nonempty zero set which is not irreducible.

Remark 2.2.22. The Nullstellensatz is not, strictly speaking, a purely algebraic statement. As we have noted, as stated it also does not hold for non-algebraically-closed fields. On the other hand, it is not very hard to develop an equivalent, purely algebraic statement which does hold for arbitrary fields. As noted in Example 2.2.14, the Nullstellensatz implies that the maximal ideals of A_n are all of the form $(x_1 - a_1, \dots, x_n - a_n)$, and in particular, modding out by any maximal ideal yields the field k. In addition, if $\mathfrak{p}$ is a prime ideal of A_n, and $f \in A_n$, suppose $f \in \mathfrak{m}$ for every maximal ideal $\mathfrak{m}$ of A_n containing $\mathfrak{p}$. This then implies that $f(P) = 0$ for all $P \in Z(\mathfrak{p})$, so $f \in I(Z(\mathfrak{p})) = \mathfrak{p}$. Thus, the Nullstellensatz implies:

(i) for every maximal ideal $\mathfrak{m}$ of A_n, we have $A_n/\mathfrak{m} = k$;
(ii) every prime ideal of A_n is equal to the intersection of the maximal ideals containing it.

Conversely, suppose we know these two statements. We claim that the Nullstellensatz follows. Indeed, given $I \subseteq A_n$, and $f \in A_n$ such that $f(P) = 0$ for all $P \in Z(I)$, let $\mathfrak{p} \subseteq A_n$ be a prime ideal containing I, and $\mathfrak{m} \subseteq A_n$ a maximal ideal containing $\mathfrak{p}$. Then $A_n/\mathfrak{m} = k$ by (i), so let $(a_1, \dots, a_n)$ be the images of $(x_1, \dots, x_n) \in k$; thus, the kernel of $A_n \to A_n/\mathfrak{m}$ contains $(x_1 - a_1, \dots, x_n - a_n)$, and since the latter is a maximal ideal, we conclude that $\mathfrak{m} = (x_1 - a_1, \dots, x_n - a_n)$. Let $P = (a_1, \dots, a_n)$; then $Z(\mathfrak{m}) = \{P\}$, so $I(Z(\mathfrak{m})) = I(P) = \mathfrak{m}$. Additionally, since $\mathfrak{m} \supseteq \mathfrak{p}$, we have $Z(\mathfrak{p}) \supseteq Z(\mathfrak{m}) = \{P\}$, so $f(P) = 0$ by hypothesis, and $f \in I(Z(\mathfrak{m})) = \mathfrak{m}$. Since this held for all $\mathfrak{m}$ containing $\mathfrak{p}$, we conclude by (ii) that $f \in \mathfrak{p}$. Now, in general the radical of I is the intersection of the prime ideals containing I (Proposition 1.14 of [**AM69**]), so we conclude that f is in the radical of I, as desired.

Thus, we have translated the Nullstellensatz into a purely algebraic statement. In fact, we can also see that (ii) follows from (i). First, observe that if (i) holds, then it also holds for every ring R finitely generated over k, since every such ring is a quotient of some A_n. Now, given $\mathfrak{p} \in A_n$, to prove (ii) it suffices to prove that the intersection of the maximal ideals of $R := A_n/\mathfrak{p}$ is equal to 0. Given $f \in R$ nonzero, the ring of fractions R_f contains a maximal ideal $\mathfrak{m}$. Then $\mathfrak{m} \cap R$ is a prime ideal not containing f, so it suffices to see that it is maximal. But $R_f \cong R[x]/(xf - 1)$ is finitely generated over R, and hence over k, so by (i), we have $R_f/\mathfrak{m} = k$. $R/(\mathfrak{m} \cap R)$ is a subring of $R_f/\mathfrak{m}$, but still contains k, so we conclude that $R/(\mathfrak{m} \cap R) = k$, and $\mathfrak{m} \cap R$ is maximal, as desired.

To summarize, we have seen that in fact, the Nullstellensatz is equivalent to the algebra statement that for any maximal ideal $\mathfrak{m}$ of A_n, we have $A_n/\mathfrak{m} = k$. We conclude by briefly discussing the generalization to arbitrary fields. Obviously, if k is not algebraically closed, it is not true that $A_n/\mathfrak{m} = k$ for all maximal ideals $\mathfrak{m}$; in fact, every finite field extension of k occurs in this fashion.

However, it turns out to be true that A_n/fm is always a finite extension of k. Equivalently, if a field extension of k is transcendental, it cannot be finitely generated as a ring. This is intuitively sensible, since if t is transcendental over k, there are infinitely many irreducible polynomials in $k[t]$, and making them all invertible should require infinitely many inverses. In fact, one can make this into a proof (see Proposition 7.9 of [**AM69**]). Finally, we mention that while our above proof that (i) implies (ii) is worded for the algebraically closed case, the argument can be adapted to the general setting, so (ii) holds as stated for arbitrary fields.

2.3. Noetherian spaces

We now further investigate the basic properties of the Zariski topology.

Definition 2.3.1. A topological space X is **Noetherian** if every descending chain

$$Y_1 \supseteq Y_2 \supseteq Y_3 \supseteq \dots$$

of closed subsets stabilizes.

This is an absurd condition, for any reasonable topological space. However, algebraic varieties with the Zariski topology are not reasonable!

The following proposition is clear.

Proposition 2.3.2. *A subspace of a Noetherian space is Noetherian.*

Remark 2.3.3. Note that the definition of Noetherian immediately implies that if X is Noetherian, it must be compact. Combining this with Proposition 2.3.2, we see that every open subset of a Noetherian space is also compact, amply demonstrating the pathology of such spaces!

We now observe:

Proposition 2.3.4. *$\mathbb{A}^n_k$ is Noetherian.*

Proof. Given $Y_1 \supseteq Y_2 \supseteq \dots$, we then have $I(Y_1) \subseteq I(Y_2) \subseteq \dots$. Then because A_n is a Noetherian ring by the Hilbert basis theorem (Theorem 2.2.1), we know that the chain of ideals must stabilize, so by Theorem 2.2.13, the original chain stabilizes as well. ☐

Corollary 2.3.5. *Any algebraic set is Noetherian.*

Proposition 2.3.6. *Any Noetherian space X can be represented uniquely as a finite union $Y_1 \cup \dots \cup Y_n$ of irreducible closed subsets, with no Y_i contained in any other.*

Proof. In fact, it is easier to prove that every closed subset of X can be written in the desired way. Let S be the set of closed subsets of X which cannot be written as a finite union of closed irreducible subsets; we will show that S is empty. If S is nonempty, then because X is Noetherian, it must contain a minimal element, say Z. Then Z cannot be irreducible, so we can write $Z = Z_1 \cup Z_2$, with $Z \neq Z_i$ for $i = 1, 2$. By minimality of Z, each of the Z_i is a finite union of irreducible closed subsets, but then so is Z, which is a contradiction. Thus, we must have that S is empty, and X can be written as a finite union of closed irreducible subsets. It is clear that we can require that none of the Y_i are contained in any other, simply by removing any which are.

It remains to show uniqueness, which we argue similarly: let Z be a minimal closed subset of X such that

$$Z = Y_1 \cup \dots \cup Y_m = Y'_1 \cup \dots \cup Y'_{m'}$$

for some closed irreducible Y_i and Y'_i, not all the same, and with no containments on either side. Then $Y_1 = (Y_1 \cap Y'_1) \cup \dots \cup (Y_1 \cap Y'_{m'})$, so by irreducibility, we have $Y_1 = Y_1 \cap Y'_i$ for some i, and therefore $Y_1 \subseteq Y'_i$. Arguing similarly, we have $Y'_i \subseteq Y_j$ for some j, so $Y_1 \subseteq Y_j$, and by hypothesis we have $j = 1$, and $Y'_i = Y_1$. But then if we set $Z' = Y_2 \cup \dots \cup Y_m =$, we see that also $Z' = \bigcup_{\ell \neq i} Y'_i$, since both can be expressed as the closure of the complement of $Y_1 = Y_i'$ in Z. It then follows from irreducibility of Y_1 that $Y_1 \not\subseteq Z'$, so we have $Z' \subsetneq Z$, contradicting minimality of Z. $\square$

DEFINITION 2.3.7. The Y_i of Proposition 2.3.6 are the **irreducible components** of X.

REMARK 2.3.8. It follows that a Noetherian space can be Hausdorff if and only if it is a finite union of points.

COROLLARY 2.3.9. *Every algebraic set is (uniquely) a finite union of varieties.*

This is the reason for the convention of restricting to the irreducible case in many situations.

EXAMPLE 2.3.10. Returning to $Z(xy) \subseteq \mathbb{A}^2_k$, we see that the irreducible components are $Z(x)$ and $Z(y)$, each of which is a variety (since x and y are irreducible).

2.4. Dimension

One of the advantages of working with Noetherian spaces is that one can use a very elementary notion of dimension.

DEFINITION 2.4.1. The **dimension** of X is the supremum of all m such that there exists a chain $Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_m$ of irreducible closed subsets of X.

The intuition is that the any proper closed subvariety of a variety should have strictly smaller dimension, and conversely it should always be possible to find a subvariety with dimension exactly 1 less. In particular, an algebraic curve is a variety whose only (proper) closed subvarieties are points.

REMARK 2.4.2. Although this definition makes sense for any topological space, it is only a reasonable notion when X is Noetherian, or in similar circumstances. For instance, the dimension (in this sense) of $\mathbb{R}^n$ in the Euclidean topology is 0.

Even when X is Noetherian, the dimension need not be finite: although any descending chain of closed subsets must be finite, there could be an infinite ascending chain. Furthermore, even if there is no infinite chain, as is the case for natural Noetherian spaces arising in algebraic geometry, there could be infinitely many incomparable chains, of arbitrarily long length.

EXAMPLE 2.4.3. Using our prior description of its topology (Example 2.1.13), the dimension of $\mathbb{A}^1$ is 1.

The definition of dimension, while easy to state, is not so easy to use directly. We therefore develop the connection to algebra, starting with a parallel definition of dimension for rings:

DEFINITION 2.4.4. The (Krull) dimension of a ring R is the supremum of all m such that there exists a chain

$$\mathfrak{p}_0 \supsetneq \mathfrak{p}_1 \supsetneq \cdots \supsetneq \mathfrak{p}_m$$

of prime ideals in R.

From our inclusion-reversing bijection between varieties and prime ideals, we immediately see:

COROLLARY 2.4.5. *If $Y \subseteq \mathbb{A}^n_k$ an algebraic set, we have $\dim Y = \dim A(Y)$.*

We now make a related pair of definitions:

DEFINITION 2.4.6. The **codimension** of an irreducible closed subset $Y \subseteq X$, denoted $\operatorname{codim}_X Y$, is the supremum over all m such that there exists a chain $Y \subseteq Y_0 \subsetneq Y_1 \subsetneq \cdots \subsetneq Y_m$ of irreducible closed subsets of X.

Definition 2.4.7. The **height** of a prime ideal $\mathfrak{p}$ in a ring R is the supremum over all m such that there exists a chain $\mathfrak{p} \supseteq \mathfrak{p}_0 \supsetneq \mathfrak{p}_1 \supsetneq \mathfrak{p}_m$ of prime ideals in R.

As with dimension, we see:

Corollary 2.4.8. *Given algebraic sets $Y \subseteq X$, with Y irreducible, the codimension of Y in X is equal to the height of $I(Y)/I(X)$ in $A(X)$.*

Remark 2.4.9. It is not clear that $\dim Y + \operatorname{codim}_X Y = \dim X$, since in principle the longest chain in X might not have Y as one of the elements in it.

For classical algebraic geometry, the fundamental theorem on dimension is the following algebra fact:

Theorem 2.4.10. *Let k be field, and R an integral domain which is a finitely-generated k-algebra. Then:*

(a) *The dimension of R is equal to the transcendence degree over k of the fraction field $K(R)$.*
(b) *For any prime ideal $\mathfrak{p} \subseteq R$, we have*

$$\operatorname{height} \mathfrak{p} + \dim R/\mathfrak{p} = \dim R.$$

See Theorem A (§13.1) and Corollary 13.4 of [**Eis95**].
We immediately conclude:

Corollary 2.4.11. $\dim \mathbb{A}^n_k = n$.

Corollary 2.4.12. *Given $Y \subseteq X$ affine varieties,*

$$\dim Y + \operatorname{codim}_X Y = \dim X.$$

Note that without Theorem 2.4.10, even the finiteness of dimension is not obvious. In particular, it is not a general consequence of Noetherianness – see Remark 4.2.9.
We also have:

Proposition 2.4.13. *If X is a topological space and $Z \subseteq X$ an irreducible closed subset, then for any open subset $U \subseteq X$ such that $U \cap Z \neq \emptyset$, we have $\operatorname{codim}_X Z = \operatorname{codim}_U Z \cap U$.*

Proof. It is clear that intersecting with U and taking closures in X creates a bijection between chains of irreducible closed subsets of X containing Z, and chains of irreducible closed subsets of U containing $Z \cap U$. Thus, the codimensions agree. □

The proof of Proposition 2.4.13 may look like it could also show that $\dim U = \dim X$, but this is not true in general, because the longest chains of irreducible closed subsets in X might contain subsets in the complement of U. However, in the case of interest to us this will not occur:

Corollary 2.4.14. *If X is an affine variety and $U \subseteq X$ a nonempty open subset, then $\dim U = \dim X$.*

Proof. Given $P \in U$, Proposition 2.4.13 implies that $\operatorname{codim}_U P = \operatorname{codim}_X P$. But by Corollary 2.4.12, $\operatorname{codim}_X P = \dim X$, and by definition $\dim U = \sup_{P \in U} \operatorname{codim}_P U$, so we conclude $\dim U = \dim X$. □

Another fundamental algebra theorem relating to dimension is the following:

Theorem 2.4.15 (Krull principal ideal). *Let R be a Noetherian ring, and $f \in R$ an element which is not a unit or a zero divisor. Then every minimal prime ideal containing f has height* 1.

More generally, if an ideal $I \subseteq R$ is generated by elements $(f_1, \ldots, f_m)$, then every minimal prime ideal containing I has height at most m.

See Theorem 10.2 of [**Eis95**].

Geometrically, this says:

COROLLARY 2.4.16. *Given $f_1, \ldots, f_m \in A_n$, and any algebraic set $Y \subseteq \mathbb{A}_k^n$, for any irreducible component Z of $Y \cap Z(f_1, \ldots, f_m)$, we have*

$$\operatorname{codim}_Y Z \leqslant m.$$

Thus, every time we add an equation, the codimension can go up by at most 1 (or equivalently, the dimension can go down by at most 1). Of course, the change will be exactly 1 unless the new polynomial vanishes on an irreducible component.

It is then natural to wonder whether, conversely, if we have a subvariety of codimension m, it can be defined by some m equations. This is true in one special case:

PROPOSITION 2.4.17. *If R is a (Noetherian) unique factorization domain, every ideal of height 1 in R is principal.*

In particular, we see:

COROLLARY 2.4.18. *If $Y \subseteq \mathbb{A}_k^n$ is a variety of codimension 1, then $Y = Z(f)$ for some $f \in A_n$.*

However, once $\operatorname{codim}_{\mathbb{A}_k^n} Y = m \geqslant 2$, it is not always possible to define Y by m equations.

EXAMPLE 2.4.19. Identify $\mathbb{A}_k^6$ with the collection of 2×3 matrices over k, and let Y be the collection of matrices of rank less than or equal to 1. We know that Y is an algebraic set, because it is given by vanishing of minors: specifically, if we use variables

$$\begin{bmatrix} x_{1,1} & x_{1,2} & x_{1,3} \\ x_{2,1} & x_{2,2} & x_{2,3} \end{bmatrix},$$

then

$$Y = Z(x_{1,1}x_{2,2} - x_{1,2}x_{2,1},\ \ x_{1,1}x_{2,3} - x_{1,3}x_{2,1},\ \ x_{1,2}x_{2,3} - x_{1,3}x_{2,2}).$$

Now, we will see in Exercise 3.2.20 below that Y is in fact irreducible of dimension 4, but that $I(Y)$ cannot be generated by any two elements.

In fact, it turns out that Y cannot be written as the zero set of any two polynomials (equivalently, that no ideal whose radical is $I(Y)$ can be generated by two elements). This last result is harder (see Example 5.6 of [**Hun07**]), but to illustrate what happens, if we just consider $Z(x_{1,1}x_{2,2} - x_{1,2}x_{2,1},\ x_{1,1}x_{2,3} - x_{1,3}x_{2,1})$, then this also has dimension 4, but in addition to Y, it contains all matrices with $x_{1,1} = x_{2,1} = 0$.

This is typical: a given variety Y of codimension m may require more than m equations to define, in which case any choice of m equations which are satisfied on Y will have a reducible zero set which contains Y but also contains other irreducible components.

EXERCISE 2.4.20. We can now prove that certain natural sets aren't algebraic, even over non-algebraically closed fields.

(a) Show that neither branch of the real hyperbola $xy = 1$ in $\mathbb{A}_{\mathbb{R}}^2$ is an algebraic set (that is, there does not exist a collection of real polynomials whose common real zero set is a branch of $xy = 1$).

(b) The real points of $y^2 = x^3 - x^2 \subseteq \mathbb{A}_{\mathbb{R}}^2$ have two connected components. Describe them, and show that one of them is algebraic, while the other isn't.

CHAPTER 3

Regular functions and morphisms

We have discussed affine varieties and some of their basic properties, but our next task is to study maps between them. In particular, we will obtain a concept of isomorphism of affine varieties, which allows us to consider them independently from their imbedding in affine space. Following Hartshorne [**Har77**], we will first consider the notion of regular functions, and then use them to define morphisms.

3.1. Regular functions

A modern perspective on geometry is that geometry is captured by considering spaces together with a designated class of maps between them. Taking this further, one can determine the maps of interest simply by designating a particular class of *functions* of interest. This leads us to start our discussion by defining a class of functions of interest. Ultimately, we will be interested in functions not only on affine varieties, but on open subsets of them.

We have already indicated that polynomials define a natural class of algebraic functions. More generally, we could consider functions defined by quotients of polynomials, wherever the denominator is nonvanishing. As with notions such as continuity and differentiability, we want our class of functions to be locally determined, which leads us to the following definition.

Definition 3.1.1. Let $Y \subseteq \mathbb{A}^n_k$ be an affine algebraic set, and $U \subseteq Y$ an open subset. A function $f : U \to k$ is **regular** if for every $P \in U$, there exists an open neighborhood $V \subseteq U$ of P such that $f = \frac{g}{h}$, where $g, h \in A_n$ and $Z(h) \cap V = \emptyset$.

Denote by $\mathscr{O}(U)$ the ring of regular functions on U.

Note that $\mathscr{O}(U)$ forms a ring just because we can add and multiply functions as usual, by adding and multiplying their values.

Example 3.1.2. If $f \in A_n$, then f defines a regular function on Y for any algebraic set $Y \subseteq \mathbb{A}^n_k$. In fact, we see that $A(Y)$ injects into $\mathscr{O}(Y)$.

Example 3.1.3. Let $Y = Z(wx - yz) \subseteq \mathbb{A}^4_k$, and let $U = Y \smallsetminus Z(x, y)$. We can define a regular function f on U by defining it to be $\frac{w}{y}$ on the complement of $Z(y)$, and to be $\frac{z}{x}$ on the complement of $Z(x)$. These functions agree on the complement of $Z(x) \cup Z(y)$, so we get a regular function.

The example shows that it if a regular function is defined on an open set U, it may not be possible to give a single expression for it which is defined on all of U. Put differently, the V in the definition may be necessary. However, the fundamental theorem on regular functions will assert in particular that $A(Y) = \mathscr{O}(Y)$, so that this doesn't occur when a regular function is defined on all of Y. Before proving this, we explore two more basic properties of regular functions.

Lemma 3.1.4. *A regular function $f : U \to k$ defines a continuous map $U \to \mathbb{A}^1_k$.*

Proof. By our description of the closed subsets of $\mathbb{A}^1_k$, it is enough to see that the preimage of a point is closed. A point in $\mathbb{A}^1_k$ is determined by some $c \in \mathbb{A}^1_k$. To check that $f^{-1}(c)$ is closed, it is enough to check on an open cover of U. Given $P \in U$, let $V \subseteq U$ be as in the definition of a regular function, so that $f = \frac{g}{h}$ on V. Then on V, we see that $f^{-1}(c)$ is the zero set of $\frac{g}{h} - c = \frac{g-ch}{h}$, which is just $Z(g - ch)$, since h is nonvanishing on V. Thus, $f^{-1}(c)$ is closed in V by definition of the Zariski topology. Since the choices of V cover U, we conclude that $f^{-1}(c)$ is closed, as desired. $\square$

An important corollary is that regular functions are much more rigid than the functions of interest in differential geometry:

Corollary 3.1.5. *Let Y be a variety, and $V \subseteq U \subseteq Y$ nonempty open subsets. If f, g are regular functions on U, and $f|_V = g|_V$, then $f = g$.*

Proof. By continuity, the locus in U on which f and g agree – which is just $(f - g)^{-1}(0)$ – is closed. But since Y is irreducible, V is dense, and we conclude that $f = g$ on all of U. $\square$

We'll need one more definition in order to prove our main theorem.

Definition 3.1.6. If $Y \subseteq \mathbb{A}^n_k$ is an affine algebraic set, and $P \in Y$, then the **local ring** at P on Y, denoted $\mathscr{O}_{P,Y}$, is the set of pairs (U, f) with $P \in U \subseteq Y$, and $f : U \to k$ regular, modulo the equivalence relation $(U, f) \sim (V, g)$ if there exists an open neighborhood W of P contained in $U \cap V$ such $f|_W = g|_W$.

If further Y is a variety, the **function field** of Y (or **field of rational functions** on Y), denoted $K(Y)$, is the set of pairs (U, f) over all $U \subseteq Y$, modulo the relation that $(U, f) \sim (V, g)$ if $f|_{U\cap V} = g|_{U\cap V}$.

Remark 3.1.7. We define the ring operations on $\mathscr{O}_{P,Y}$ and $K(Y)$ using restrictions. In the case of function fields, we use that $U \cap V$ is always nonempty, because Y is irreducible. If Y is irreducible, then Corollary 3.1.5 implies that we could have defined our equivalence relation for $\mathscr{O}_{P,Y}$ just as we did for $K(Y)$.

The following is straightforward from the definitions:

Proposition 3.1.8. *For an algebraic set Y and $P \in Y$, the local ring $\mathscr{O}_{P,Y}$ is a local ring (i.e., has a unique maximal ideal), and its maximal ideal consists of (U, f) with $f(P) = 0$. In addition, $K(Y)$ is a field.*

Remark 3.1.9. Again assuming Y is a variety, we have natural injections

$$A(Y) \hookrightarrow \mathscr{O}(Y) \hookrightarrow \mathscr{O}_{P,Y} \hookrightarrow K(Y),$$

with the last two being a consequence of Corollary 3.1.5, so we can think of everything as being inside of $K(Y)$.

If Y is not irreducible, we still have an inclusion $A(Y) \hookrightarrow \mathscr{O}(Y)$ and a natural map $\mathscr{O}(Y) \to \mathscr{O}_{P,Y}$, but the latter will not be injective for most choices of P.

Exercise 3.1.10. If $X \subseteq \mathbb{A}^n_k$ is an algebraic set, and $P \in X$, describe the kernel of $\mathscr{O}(X) \to \mathscr{O}_{P,X}$ in terms of restrictions to irreducible components of X.

Theorem 3.1.11. *Given $Y \subseteq \mathbb{A}^n_k$ an affine variety, we have:*

(a) *$\mathscr{O}(Y) = A(Y)$;*
(b) *for all $P \in Y$, we have $\mathscr{O}_{P,Y} = A(Y)_{\mathfrak{m}_P}$, where $\mathfrak{m}_P$ denotes the maximal ideal of polynomials vanishing at P;*
(c) *$K(Y)$ is the fraction field of $A(Y)$.*

In the above, $A(Y)_{\mathfrak{m}_P}$ denotes the localization of $A(Y)$ at $\mathfrak{m}_P$; since $A(Y)$ is an integral domain, this can be interpreted simply as the subring of the fraction field of $A(Y)$ consisting of elements whose denominators are not in $\mathfrak{m}_P$.

Proof. We first prove (b). Given $P \in Y$, and $f = \frac{g}{h} \in A(Y)_{\mathfrak{m}_P}$, with $h \notin \mathfrak{m}_P$, then f also gives an element of $\mathscr{O}_{P,Y}$, and it is straightforward to check that we obtain a map $A(Y)_{\mathfrak{m}_P} \to \mathscr{O}_{P,Y}$, which is moreover injective, since we can think of both rings as subrings of $K(Y)$. Furthermore, the map is clearly surjective by the definition of regular functions, recalling that in $\mathscr{O}_{P,Y}$ we can always restrict to a neighborhood of P on which f is defined by a single quotient of polynomials. We thus conclude (b).

Now, if $K(A(Y))$ is the field of fractions of $A(Y)$, then for any $P \in Y$, we have $K(A(Y)) = K(A(Y)_{\mathfrak{m}_P}) = K(\mathscr{O}_{P,Y})$, by (b). So $K(A(Y)) \supseteq \cup_{P \in Y} \mathscr{O}_{P,Y}$, but every element of $K(Y)$ is in some $\mathscr{O}_{P,Y}$, and we conclude (c).

Finally, $\mathscr{O}(Y) \subseteq \cap_{P \in Y} \mathscr{O}_{P,Y}$, so by the fact that any integral domain is equal to the intersection of its localizations at all maximal ideals, we have

$$A(Y) \subseteq \mathscr{O}(Y) \subseteq \bigcap_{\mathfrak{m}} A(Y)_{\mathfrak{m}} = A(Y).$$

This proves (a), completing the proof of the theorem. $\square$

We thus see that while a regular function on a general open subset U may need more than one expression to define on all of U, if it is defined on all of Y, it can necessarily be given by a single polynomial. In particular, this now justifies and makes precise our earlier assertion that we can consider the ring $A(Y)$ to be the ring of algebraic functions on Y.

It is helpful to understand what happens to ideals of rings when we invert some elements, as in constructing $A(Y)_{\mathfrak{m}}$ from $A(Y)$.

Exercise 3.1.12. Let R be a ring, and $S \subseteq R$ a subset which is closed under multiplication. Let $S^{-1}R$ be the ring obtained by inverting elements of S, and $f : R \to S^{-1}R$ the natural homomorphism. Then:

(a) If $I \subseteq S^{-1}R$ is an ideal, then the ideal generated by $f(f^{-1}(I))$ is equal to I; in particular, $I \mapsto f^{-1}(I)$ induces an injection from ideals of $S^{-1}R$ to ideals of R.
(b) The injection from (a) induces a bijection between prime ideals of $S^{-1}R$ and prime ideals of R disjoint from S.

Corollary 3.1.13. *If $Y \subseteq \mathbb{A}^n_k$ is an affine variety, then* $\dim \mathscr{O}_{P,Y} = \dim Y$*, and $K(Y)$ is a field finitely generated over k, of transcendence degree equal to* $\dim Y$*.*

Proof. This is an immediate consequence of the theorem and our previous results on dimension, with the added observation that the height of $\mathfrak{m}_P$ in $A(Y)_{\mathfrak{m}_P}$ is equal to the height in $A(Y)$, so we have that

$$\dim A(Y)_{\mathfrak{m}_P} = \operatorname{height} \mathfrak{m}_P = \dim A(Y) - \dim A(Y)/\mathfrak{m}_P = \dim A(Y) = \dim Y.$$

$\square$

We can generalize the previous corollary to the setting of affine algebraic sets with an additional definition.

Definition 3.1.14. If $X \subseteq \mathbb{A}^n_k$ is an affine algebraic set, and $P \in X$, define the **dimension** of X at P, denoted $\dim_P X$, to be the maximum dimension of the irreducible components of X containing P.

Exercise 3.1.15. Let $X \subseteq \mathbb{A}^n_k$ be an affine algebraic set, and $P \in X$. Then:

(a) $\dim_P X$ is equal to the minimum dimension of all open neighborhoods of P in X.
(b) $\dim_P X = \dim \mathscr{O}_{P,X}$.

3.2. Morphisms

We are now in a good position to define and study morphisms. The idea is surprisingly simple, and should make clear the idea that in order to describe a class of maps of interest, it is often enough to describe a class of functions of interest.

Before going further, we will broaden our class of varieties slightly, so that our definition will be in suitable generality.

DEFINITION 3.2.1. A **quasiaffine** variety is an open subset of an affine variety.

Our definition of regular function was stated in the context of quasiaffine varieties, missing only the terminology. In particular, the notation $\mathscr{O}(Y)$ makes sense for a quasiaffine variety (however, for reasons which will soon become clear, we reserve the notation $A(Y)$ for the affine case).

Until now, we have only considered closed subvarieties. But now that we have introduced quasiaffine varieties, we change our terminology as follows:

DEFINITION 3.2.2. Let X be a quasiaffine variety. A **closed subvariety** of X is a closed, irreducible subset of X. An **open subvariety** of X is an open subset of X. A **subvariety** X is an open subvariety of a closed subvariety of X.

Note that we could have equivalently defined a subvariety to be a closed subvariety of an open subvariety, or the intersection of an open and a closed subvariety.

The definition of morphism is then as follows.

DEFINITION 3.2.3. Given X, Y quasiaffine varieties, a **morphism** $\varphi : X \to Y$ is a continuous map which satisfies the condition that for every $V \subseteq Y$ open, and every regular function f on V, the composition $f \circ \varphi : \varphi^{-1}(V) \to k$ is regular.

A morphism φ is an **isomorphism** if it is has an inverse which is also a morphism.

REMARK 3.2.4. It is clear that a composition of morphisms is a morphism, so we obtain a category of quasiaffine varieties, with the affine varieties as a full subcategory.

It is clear from the definition that a morphism $X \to Y$ induces a ring homomorphism $\mathscr{O}(Y) \to \mathscr{O}(X)$. We will see that when X and Y are affine, we can go in the other direction as well, but that this doesn't work if X and Y are only assumed quasiaffine.

The prototypical example of a morphism, which guides our intuition for morphisms, is the following:

EXAMPLE 3.2.5. Given quasiaffine varieties $X \subseteq \mathbb{A}^n_k$ and $Y \subseteq \mathbb{A}^m_k$, and polynomials $f_1, \dots, f_m \in A_n$, if the map $(f_1, \dots, f_m) : \mathbb{A}^n_k \to \mathbb{A}^m_k$ maps X into Y, then the induced map $X \to Y$ is a morphism.

We also see that we can reinterpret regular functions in terms of morphisms.

EXAMPLE 3.2.6. Regular functions on a quasiaffine variety X are the same as morphisms $X \to \mathbb{A}^1_k$.

Both of the above examples can be deduced from the following.

EXERCISE 3.2.7. Show the following.

(a) If X is a quasiaffine variety, and a map $\varphi : X \to \mathbb{A}^m_k$ is given by functions $(f_1, \dots, f_m)$, then φ is a morphism if and only if all the f_i are regular functions.

(b) If $\varphi : X \to Y$ is a morphism of quasiaffine varieties, and $Z \subseteq Y$ is a subvariety, then φ factors through Z if and only if the image of φ is contained in Z.

It may be the case that a proper open subset of an affine variety is isomorphic to a different affine variety.

Example 3.2.8. Let $U = \mathbb{A}^1_k \smallsetminus \{(0)\}$. We claim U is isomorphic to $Y = Z(xy-1) \subseteq \mathbb{A}^2_k$. This is easily verified using Exercise 3.2.7. In one direction, the projection map $(x,y) \mapsto (x)$ gives a morphism $Y \to \mathbb{A}^1_k$, which has image equal to U, and therefore gives a morphism $Y \to U$. The inverse is given by the formula $(x) \mapsto (x, 1/x)$; since x and $1/x$ are regular functions on U, the inverse is likewise a morphism.

More generally, we have the following.

Proposition 3.2.9. *Let Y be an affine variety, and $f \in A(Y)$. Then $Y_f := Y \smallsetminus Z(f)$ is isomorphic to an affine variety, and $A(Y_f) = A(Y)_f$.*

Proof. Suppose $Y \subseteq \mathbb{A}^n_k$, so that $I(Y) \subseteq A_n$, and choose a representative for f in A_n; then we claim more specifically that Y_f is isomorphic to $Z(J) \subseteq \mathbb{A}^{n+1}_k$, where $J = (I(Y), tf-1) \subseteq A_n[t]$. This again follows from Exercise 3.2.7, since the projection map $\mathbb{A}^{n+1}_k \to \mathbb{A}^n_k$ which drops the coordinate t maps $Z(J)$ onto Y_f, and therefore induces a morphism $Z(J) \to Y_f$, and the inverse is given by

$$(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, 1/f(x_1, \ldots, x_n)),$$

which is given by regular functions on Y_f, and therefore is likewise a morphism. Finally, it is clear that the given isomorphism identifies $A(Z(J))$ with $A(Y)_f$, as desired. □

A surprising consequence of this proposition is that quasiaffine varieties can always be covered by varieties which are isomorphic to affine varieties.

Corollary 3.2.10. *Let Y be a quasiaffine variety. Then open subvarieties of Y which are isomorphic to affine varieties form a base of the topology of Y, and in particular cover Y.*

Proof. Since a quasiaffine variety is an open subset of an affine variety, it is enough to prove the statement for affine varieties. By Proposition 3.2.9, it is enough to prove that if Y is affine, then the subsets Y_f form a base for the topology. Say $Y \subseteq \mathbb{A}^n_k$, so that $I(Y) \subseteq A_n$. Given $P \in Y$, and $U \subseteq Y$ an open neighborhood of P, let $Z = Y \smallsetminus U$; then $Z = Z(I)$ for some ideal I, and by definition, there is some $f \in I$ such that $f(P) \neq 0$. Then $Z(f) \supseteq Z(I)$, so $Y_f \subseteq U$, as desired. □

We now specialize to affine varieties, where Example 3.2.5, together with Theorem 3.1.11, will imply that morphisms correspond to ring maps. More specifically, it is important to keep track of k, so we observe that $\mathscr{O}(X)$ for any quasiaffine variety, and $A(X)$ for an affine variety, are both naturally k-algebras, and that if we have a morphism $X \to Y$, the induced map $\mathscr{O}(Y) \to \mathscr{O}(X)$ is a homomorphism of k-algebras. We then have:

Corollary 3.2.11. *Given a quasiaffine variety X and an affine variety Y, composition induces a bijection between morphisms $X \to Y$ and k-algebra homomorphisms $A(Y) \to \mathscr{O}(X)$.*

Proof. First recall that we observed earlier that a morphism $X \to Y$ gives a k-algebra homomorphism $\mathscr{O}(Y) \to \mathscr{O}(X)$, and hence a k-algebra homomorphism $A(Y) \to \mathscr{O}(X)$.

In the other direction, say $A(Y) = A_m/I(Y)$. Then a k-algebra homomorphism $A(Y) \to \mathscr{O}(X)$ can be lifted to a homomorphism $A_m \to \mathscr{O}(X)$, which is described by an m-tuple of regular functions $f_1, \ldots, f_m \in \mathscr{O}(X)$, by letting f_i be the image of x_i. Then since the f_i came from a homomorphism $A(Y) \to \mathscr{O}(X)$, they must map $I(Y)$ to 0, which implies (using that Y is closed) that the induced map $X \to \mathbb{A}^m_k$ has image contained in Y. Thus, by Exercise 3.2.7, we get a morphism $X \to Y$.

It is straightforward to check that these constructions are inverse to one another, proving the corollary. □

Remark 3.2.12. After we have introduced abstract varieties, we will see that Corollary 3.2.11 holds even when X is an abstract variety.

Using that the association between morphisms and k-algebra homomorphisms commutes with composition, we can conclude the following.

EXERCISE 3.2.13. If X, Y are affine varieties, and $\varphi : X \to Y$ is a morphism, let $\varphi^* : A(Y) \to A(X)$ be the corresponding k-algebra homomorphism. Then φ is an isomorphism if and only if φ^* is an isomorphism.

In fancier categorical language, we have:

COROLLARY 3.2.14. *The map $X \mapsto A(X)$, together with the identification $A(X) = \mathscr{O}(X)$, induces an arrow-reversing equivalence of categories between the category of affine varieties, and the category of k-algebras which are integral domains and finitely generated over k.*

This is immediate from Corollary 3.2.11 and Remark 2.2.19, taking into account Theorem 3.1.11 (a).

REMARK 3.2.15. Given the abstract "dots and arrows" nature of categories, it might seem as though we lose a lot of information when we forget the structure of affine varieties as having come from subsets of $\mathbb{A}^n_k$, and only consider the category. However, the category contains a surprisingly rich amount of information – see Appendix 3.A below.

We have now seen that in the affine case, there is a very satisfying equivalence between thinking about affine varieties and their maps, and thinking about the associated affine coordinate rings. However, we can quickly see that this is not true for quasiaffine varieties.

EXERCISE 3.2.16. Show that if $U \subseteq \mathbb{A}^2_k$ is the complement of $(0,0)$, then the restriction map $\mathscr{O}(\mathbb{A}^2_k) \to \mathscr{O}(U)$ is an isomorphism.

To illustrate the power of Corollary 3.2.11, we can produce our first example of a quasiaffine variety which is not isomorphic to an affine variety.

COROLLARY 3.2.17. $U = \mathbb{A}^2_k \smallsetminus \{(0,0)\}$ *is not isomorphic to an affine variety.*

PROOF. If U were affine, then because $\mathbb{A}^2$ is affine, and the restriction map induces an isomorphism $\mathscr{O}(\mathbb{A}^2_k) \xrightarrow{\sim} \mathscr{O}(U)$, then by Corollary 3.2.11 the inclusion map $U \hookrightarrow \mathbb{A}^2_k$ would have to be an isomorphism (see Exercise 3.2.13). But the inclusion map isn't even bijective, so we conclude the desired statement. $\square$

REMARK 3.2.18. Note that in the corollary, it is important not just to be thinking coarsely in terms of isomorphic and non-isomorphic, but to be keeping track of what happens with a single map. Otherwise, we start from the fact that $\mathscr{O}(\mathbb{A}^2_k)$ is isomorphic to $\mathscr{O}(U)$, and decide that if we want to show that U isn't affine, we need to prove that it isn't isomorphic to $\mathbb{A}^2_k$, but this isn't much easier to do directly than showing that U isn't isomorphic to any affine variety. In contrast, by keeping track of the map, we get a trivial proof that U isn't isomorphic to an affine variety.

Note the contrast between Corollary 3.2.17 and Proposition 3.2.9: from the latter, we conclude that removing any hypersurface (i.e., curve) from $\mathbb{A}^2_k$ will give us something still isomorphic to an affine variety, so in some sense, $\mathbb{A}^2_k \smallsetminus \{(0,0)\}$ is bad because the set we have removed is "too small."

EXERCISE 3.2.19. Show that if $\varphi : X \to Y$ is a morphism of quasiaffine varieties, and $Z \subseteq Y$ is a subvariety, then φ factors through Z if and only if the image of φ is contained in Z.

EXERCISE 3.2.20. We fill in some details of Example 2.4.19. Let $Y \subseteq \mathbb{A}^6_k$ be the collection of 2×3 matrices of rank at most 1.

(a) Show that Y is irreducible. Hint: construct a natural morphism $\mathbb{A}^4_k \to Y$, and show that the image is dense.

(b) Show that $\dim Y = 4$.
(c) Show that $I(Y)$ cannot be generated by any two polynomials.

EXERCISE 3.2.21. Let $X = Z(y^2 - (x^3 - x)) \subseteq \mathbb{Z}_k^2$. Show that X is not isomorphic to $\mathbb{A}_k^1$. Hint: which properties does $k[t]$ have which are not shared by $A(X)$?

EXERCISE 3.2.22. Let $X \subseteq \mathbb{A}_k^n$, $Y \subseteq \mathbb{A}_k^m$ be affine algebraic sets. Define the **product** $X \times Y$ to be the subset $X \times Y \subseteq \mathbb{A}_k^{n+m}$.

(a) Show that $X \times Y$ is closed in $\mathbb{A}_k^{n+m}$, and that if X and Y are varieties, then $X \times Y$ is also a variety.
(b) Show that the projection maps $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$ are morphisms, and that if Z is a quasiaffine variety, then morphisms $Z \to X \times Y$ are in natural bijection with pairs of morphisms $Z \to X, Z \to Y$ via composing with p_1 and p_2.
(c) Show that if X and Y are varieties of dimensions d_1 and d_2 respectively, then $X \times Y$ has dimension $d_1 + d_2$.
(d) Show that $A(X \times Y) = A(X) \otimes_k A(Y)$.

EXERCISE 3.2.23. According to part (b) of Exercise 3.2.22, if X is an affine variety, then the **diagonal map** $\Delta : X \to X \times X$ sending x to (x, x) is a morphism. Show that $\Delta(X)$ is closed, and Δ induces an isomorphism $X \xrightarrow{\sim} \Delta(X)$.

3.3. Rational maps

In algebraic geometry, it turns out that we frequently want to consider maps that are only defined on open subsets of the source variety. This is relevant especially in setting up the notion of birational equivalence, which is a weakening of isomorphism and arises naturally in various settings.

DEFINITION 3.3.1. If X, Y are quasiaffine varieties, a **rational map** $\varphi : X \dashrightarrow Y$ is an equivalence class of pairs (U, φ_U) where $U \subseteq X$ is a nonempty open subset, and $\varphi_U : U \to Y$ is a morphism. The equivalence relation is that $(U, \varphi_U) \sim (V, \varphi_V)$ if $\varphi_U|_{U\cap V} = \varphi_V|_{U\cap V}$.

In order to verify that the stated equivalence relation is in fact an equivalence relation, we use the following:

LEMMA 3.3.2. *If X, Y are quasiaffine varieties, and $\varphi, \psi : X \to Y$ are morphisms such that $\varphi|_U = \psi|_U$ for some nonempty open $U \subseteq X$, then $\varphi = \psi$.*

PROOF. We have $Y \subseteq \mathbb{A}_k^n$ for some n; then by composing with the inclusion, we may assume that $Y = \mathbb{A}_k^n$. But we know from Exercise 3.2.7 that morphisms $X \to \mathbb{A}_k^n$ are determined by n-tuples of regular functions on X, and if two such n-tuples agree on U, they must agree on X by Corollary 3.1.5. $\square$

The most important kind of rational maps are those with dense images. We observe:

PROPOSITION 3.3.3. *Suppose $\varphi : X \to Y$ is a morphism of quasiaffine varieties, and $U \subseteq X$ nonempty and open. Then $\varphi(X)$ is dense in Y if and only if $\varphi(U)$ is dense in Y.*

PROOF. Since $\varphi(U) \subseteq \varphi(X)$, we need to verify that if $\varphi(X)$ is dense, then $\varphi(U)$ is likewise dense. But if $V \subseteq Y$ is nonempty and open, then $\varphi^{-1}(V)$ is nonempty and open in X, hence dense by irreducibility of X. We conclude that $\varphi^{-1}(V) \cap U \neq \emptyset$, and hence that $\varphi(U)$ is dense, as desired. $\square$

This motivates the following:

DEFINITION 3.3.4. A rational map $X \dashrightarrow Y$ is **dominant** if some (equivalently, every) representative (U, φ_U) has $\varphi_U(U)$ dense in Y.

In general, it may not be possible to compose two rational maps – the image of the first might be contained in the complement of every open set on which the second is defined. However, the composition of two dominant rational maps always makes sense, and is again a dominant rational map. Thus, we can talk about the category whose objects are quasiaffine varieties, and whose morphisms consist of dominant rational maps. The isomorphisms in this category are the following:

Definition 3.3.5. A **birational** map $X \dashrightarrow Y$ is a rational map which admits a rational map inverse. We say X and Y are **birational** if there exists a birational map between them.

Here, we mean that there is a rational map $Y \dashrightarrow X$ such that the composition in each direction exists as a rational map, and is equal to the identity. This condition automatically implies dominance, so a birational map is necessarily dominant.

Example 3.3.6. For a quasiaffine variety X, a rational function on X is the same as a rational map $X \dashrightarrow \mathbb{A}^1_k$. Indeed, this follows immediately from the fact that a regular function on an open subset U of X is the same as a morphism $U \to \mathbb{A}^1_k$. A rational function corresponds to a dominant rational map if and only if it is nonconstant: although it is not immediately obvious what the image of a morphism $U \to \mathbb{A}^1_k$ could look like, its closure must be an irreducible closed subset of $\mathbb{A}^1_k$, and hence is either a single point or all of $\mathbb{A}^1_k$.

Example 3.3.7. The morphism $\mathbb{A}^2_k \to \mathbb{A}^2_k$ defined by sending (x, y) to (x, xy) is not an isomorphism, but it is a birational map. We can define an inverse sending (u, v) to $(u, v/u)$ which is defined away from $u = 0$.

Example 3.3.8. The morphism $\mathbb{A}^1_k \to \mathbb{A}^2_k$ defined by $(t) \mapsto (t^2, t^3)$ has image $Z = Z(y^2 - x^3)$. It is clear that Z is a curve, so since both $\mathbb{A}^1_k$ and Z have the cofinite topology, the induced morphism $\mathbb{A}^1_k \to Z$ is a homeomorphism. This is not an isomorphism, though: the induced map is $k[x, y]/(y^2 - x^3) \hookrightarrow k[t]$ sending x to t^2 and y to t^3, and the image of this map is $k[t^2, t^3] \subseteq k[t]$, which is not all of $k[t]$.

On the other hand, we do see that the morphism gives a birational map $\mathbb{A}^1_k \to Z$, since the inverse can be defined by $(x, y) \mapsto (y/x)$ on the complement of $x = 0$ in Z.

The main observation is then that given a dominant rational map $\varphi : X \dashrightarrow Y$ and $f \in K(Y)$ a rational function, composition induces a rational function $f \circ \varphi \in K(X)$. This operation commutes with addition and multiplication of rational functions, and sends constant functions to (the same) constants, so we obtain a homomorphism $\varphi^* : K(Y) \to K(X)$ of fields over k. Since every field homomorphism is an injection, this realizes $K(Y)$ as a subfield of $K(X)$.

Theorem 3.3.9. *For any two quasiaffine varieties X, Y, the map $\varphi \mapsto \varphi^*$ induces a bijection between the set of dominant rational maps $X \dashrightarrow Y$ and the set of k-algebra homomorphisms $K(Y) \hookrightarrow K(X)$.*

Proof. Starting from a homomorphism $\theta : K(Y) \hookrightarrow K(X)$, we have to construct a dominant rational map $X \dashrightarrow Y$. For this purpose, we can freely restrict X or Y to open subsets, so we might as well assume they are both affine. Then we know that $K(Y) = K(A(Y))$ and $K(X) = K(A(X))$. Let $y_1, \ldots, y_n$ be generators for $A(Y)$ over k. Then $\theta(y_i) \in K(X)$, so we can write $\theta(y_i) = \frac{x_i}{h_i}$ for some $x_i, h_i \in A(X)$, with $h_i \neq 0$. Set $U = X_{h_1 \cdots h_n} \subseteq X$; then U is still an affine variety, and we have $A(U) = A(X)_{h_1 \cdots h_n}$ and $K(U) = K(X)$, so we can think of θ as mapping $A(Y)$ into $A(U)$. This then gives us a morphism $U \to Y$, which is a representative of the desired rational map $X \dashrightarrow Y$. It is dominant because it is induced by an injective ring map. A definition chase verifies that our construction is inverse to the map $\varphi \mapsto \varphi^*$, proving the theorem. $\square$

Remark 3.3.10. $K = K(X)$ for some X if and only if K is a finitely generated field extension of k, so Theorem 3.3.9 gives an arrow-reversing equivalence between the category of quasiaffine varieties with dominant rational maps and the category of finitely generated field extensions of k.

Corollary 3.3.11. *For two quasiaffine varieties X, Y, the following are equivalent:*

(a) *X and Y are birational;*
(b) *X and Y have isomorphic (nonempty) open subsets;*
(c) *$K(X) \cong K(Y)$.*

Proof. The theorem gives the equivalence of (a) and (c). It is clear that (b) implies (a). Conversely, if $\varphi : X \dashrightarrow Y$ and $\psi : Y \dashrightarrow X$ are inverse to one another, represented by (U, φ_U) and (V, ψ_V), then $\psi \circ \varphi$ is represented by $(\varphi_U^{-1}(V), \psi_V \circ \varphi_V)$ and $\varphi \circ \psi$ is represented by $(\psi_V^{-1}(U), \varphi_U \circ \psi_V)$, and by hypothesis both of these are equal to the identity. Then we see that φ_U and ψ_V induces mutually inverse morphisms between $\varphi_U^{-1}(\psi_V^{-1}(U))$ and $\psi_V^{-1}(\varphi_U^{-1}(V))$, yielding (b). $\square$

Remark 3.3.12. Birationality is not an interesting property in most geometric categories: for instance, any two differentiable manifolds of the same dimension have many diffeomorphic open subsets, and more specifically every open point has a neighborhood diffeomorphic to an open subset of $\mathbb{R}^n$. However, it is quite rare for a variety to have an open subset isomorphic to an open subset of $\mathbb{A}^n_k$; such varieties are called **rational**, and they are considered quite special. We will return to this in the context of curves.

Example 3.3.13. If k has characteristic $p > 0$, the **Frobenius** morphism $\mathbb{A}^1_k \to \mathbb{A}^1_k$ induced by sending t to t^p is a homeomorphism of $\mathbb{A}^1_k$ onto itself: it is bijective because pth roots are always unique in characteristic p, and have to exist over any algebraically closed field; it is then a homeomorphism because the topology is cofinite.

However, this morphism is not an isomorphism, and in fact, not even a birational map: this follows from Theorem 3.3.9 because it induces the field extension $k(t)$ over $k(t^p)$, which is a degree-p extension.

An illustration of the power of the techniques developed so far is the following:

Theorem 3.3.14. *Every variety X is birational to a hypersurface.*

Proof. Given Corollary 3.3.11, this reduces to two theorems in algebra: the first asserts that, if $K(X)$ has transcendence degree n, then $K(X)$ can be realized as a finite, separable extension of the field $k(x_1, \ldots, x_n)$ of rational functions in n variables (see Corollary A.17 of [**Eis95**]). The second is the primitive element theorem, which says that $K(X)$ can then be generated by a single element y over $k(x_1, \ldots, x_n)$ – that is, we have $K(X) \cong k(x_1, \ldots, x_n)[x_{n+1}]/(f)$ for some polynomial f with coefficients in $k(x_1, \ldots, x_n)$. We can clear denominators of the coefficients so that they lie in $k[x_1, \ldots, x_n]$, so that f can be considered as a polynomial in $x_1, \ldots, x_{n+1}$ with coefficients in k. We then have $Z(f) \subseteq \mathbb{A}^{n+1}_k$, and we conclude that X is birational to $Z(f)$, as desired. $\square$

Remark 3.3.15. Theorem 3.3.14 does not assert that X has an open cover by subvarieties isomorphic to open subsets of hypersurfaces. This would in fact be false, as will be easy to see once we have discussed singularities. It merely asserts that *some* nonempty open subset of X is isomorphic to an open subset of a hypersurface.

3.A. Recovering geometry from categories

The fundamental organizing result on affine algebraic varieties over an algebraically closed field k is surely Corollary 3.2.14, that the category they form is equivalent to the (opposite) category of finitely generated integral domains over k. This single statement encapsulates the facts that two affine varieties are isomorphic if and only if their coordinate rings are isomorphic, and more generally, that every morphism of affine varieties come from a homomorphism (in the other direction) of their coordinate rings.

However, this abstract categorical statement strips away the rich geometry enjoyed by the varieties themselves. At first, it may also seem as though abstract statements about the category cannot possibly capture the geometric content of its objects. And surely, the geometric definitions and concepts remain an important and useful foundation in the study of varieties, even as the translation to algebra provides crucial tools. Nonetheless, it is the goal of this note to demonstrate that a surprising amount of the geometry of varieties can be recovered from the abstract "dots and arrows" information encoded by the category. This underlines the power of the fundamental categorical equivalence.

Let k be an algebraically closed field. We denote by $\mathbf{Var}/k$ the category of affine algebraic varieties over k. Then the statement we will prove is the following.

PROPOSITION 3.A.1. *Let X be an object of $\mathbf{Var}/k$; that is, an affine variety. Then the set of points of X, as well as the Zariski topology on it, can be recovered completely from the data of the category $\mathbf{Var}/k$.*

We assume the reader is familiar with the definition of a category, with final and initial objects, and monomorphisms and epimorphisms. The affine variety consisting of a single point will play an important role.

NOTATION 3.A.2. Denote by $*$ the isomorphism class of a single point.

Note that any two affine varieties which are a point are isomorphic, in fact uniquely.

We first see:

LEMMA 3.A.3. *The point is the universal final object of $\mathbf{Var}/k$.*

PROOF. This is tautological if one allows $\mathbb{A}^0$ as an object of $\mathbf{Var}/k$, but even if not, it is clear that for any representative of $*$ as a subvariety of $\mathbb{A}^n$, every affine variety X has a unique function $X \to *$, which is a morphism. □

COROLLARY 3.A.4. *The set underlying an affine variety X may be recovered from $\mathbf{Var}/k$.*

PROOF. Indeed, we know from Lemma 3.A.3 that $*$ is recognizable from the structure of $\mathbf{Var}/k$. If we choose any representative of $*$, we see that the set of morphisms $* \to X$ are in bijection with the set of points of X, since for any $P \in X$, there is clearly a unique function $* \to X$ with image P, and this can be described by polynomials as a constant function, so is a morphism. □

We next address the question of recovering the topology on X. We recall a relevant definition from category theory.

DEFINITION 3.A.5. A morphism $h : A \to B$ in a category is an **extremal epimorphism** if it is an epimorphism such that if $h = f \circ g$, and f is a monomorphism, then f is an isomorphism. Similarly, h is an **extremal monomorphism** if it is a monomorphism such that if $h = f \circ g$, and g is an epimorphism, then g is an isomorphism.

We will now start working with rings as well, so let $\mathbf{Alg}/k$ denote the category of finitely generated integral domains over k.

LEMMA 3.A.6. *In the category $\mathbf{Alg}/k$, a morphism $h : A \to B$ is a monomorphism if and only if it is injective. It is an extremal epimorphism if and only if it is surjective.*

Proof. Since a ring homomorphism is determined by the map on underlying sets, it is clear that an injective map is a monomorphism, and a surjective map is an epimorphism, and indeed an extremal epimorphism. Conversely, suppose $h : A \to B$ has kernel I, and $a \in I$ is non-zero. Then consider the homomorphisms $g_1, g_2 : k[t] \to A$ determined by sending t to 0 or to a. Then $h \circ g_1 = h \circ g_2$, so h is not a monomorphism. This establishes the first assertion. On the other hand, suppose $h : A \to B$ is not surjective. Then h factors as $h : A \to h(A) \to B$, with the latter homomorphism injective, hence a monomorphism. Thus h is not an extremal epimorphism. $\square$

Remark 3.A.7. The proof shows that for h to be surjective is in fact equivalent to the condition that if $h = f \circ g$, and f is a monomorphism, then f is an isomorphism (that is, we don't need to separately require that h is an epimorphism).

We need one more definition.

Definition 3.A.8. We say that a morphism $Y \to X$ of affine varieties is a **closed imbedding** if it factors $Y \to Z \to X$ where $Y \to Z$ is an isomorphism, and Z is a closed subvariety of X.

This is now enough for us to prove the main result.

Proof of Proposition 3.A.1. Our first observation is that if X is an affine variety, and Y is another affine variety, with $f : Y \to X$ a morphism, then we can recover $f(Y) \subseteq X$ as the set of morphisms $* \to X$ which factor through Y. Next, it follows immediately from the definition of the Zariski topology and our correspondence theorems that $f : Y \to X$ is a closed imbedding if and only if $f^\sharp : A(X) \to A(Y)$ is surjective. By Lemma 3.A.6, this is equivalent to $f^\sharp$ being an extremal epimorphism. By our categorical equivalence, we see that f is a closed imbedding if and only if it is an extremal monomorphism. But this is a purely categorical condition. We can thus reconstruct the Zariski topology on X by setting the closed subsets to be finite unions of images $f(Y) \subseteq X$, where $f : Y \to X$ is any extremal monomorphism in $\mathbf{Var}/k$. $\square$

CHAPTER 4

Singularities

The topic of smoothness and singularities appears in some form in almost all algebraic geometry. The intuition is that a variety is smooth at a point if in a suitable sense it "looks like a differentiable manifold in a neighborhood of the point." One can make this into a precise (and provable) statement when working over the complex numbers.

The study of singularities is largely a local matter, so we will work throughout with affine varieties, or more generally, affine algebraic sets. Since we know that every quasiaffine variety is covered by affine varieties, this will pose no restriction in practice.

4.1. Tangent lines and singularities

We begin with some motivating examples.

EXAMPLE 4.1.1. Consider the plane curves $xy = 1$, $y^2 = x^3 - x$, $xy = 0$, and $y^2 = x^3$ shown in Figure 1.

The first two are nonsingular, while the last two have singularities at the origin. The third one is not a variety, but we could make an irreducible version with a similar picture at the origin by considering $y^2 = x^3 + x^2$.

While the picture is clear enough, it is not immediately clear how to give an actual definition of a singularity. One approach is by studying tangent spaces.

DEFINITION 4.1.2. Given a polynomial $f \in A_n$ without repeated factors, set $Z = Z(f) \subseteq \mathbb{A}^n_k$. Given also $P = (b_1, \ldots, b_n) \in Z$, a parametric line $(x_1(t), \ldots, x_n(t)) = (a_1 t + b_1, \ldots, a_n t + b_n)$ is **tangent** to Z at P if $f(a_1 t + b_1, \ldots, a_n t + b_n) \in k[t]$ vanishes to order at least 2 at $t = 0$. (Note that by construction, the line passes through P at $t = 0$, so we automatically have vanishing to order at least 1)

Although we have defined tangency of a line in terms of a parametrization, it is easy to rephrase the definition in a way which is independent of the parametrization of the line.

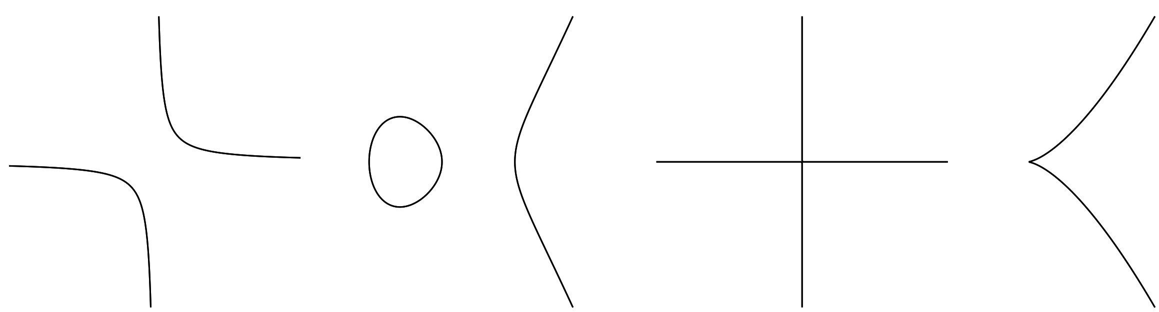

FIGURE 1. The four plane curves $xy = 1$, $y^2 = x^3 - x$, $xy = 0$, and $y^2 = x^3$.

EXAMPLE 4.1.3. Look at $(1,1)$ on the curve $f(x,y) = xy - 1 = 0$. The line $(at+1, ct+1)$ has $f(at+1, ct+1) = act^2 + (a+c)t + 1 - 1 = act^2 + (a+c)t$. We conclude that it is tangent to the curve at $(1,1)$ if and only if $a = -c$, so the tangent line is $y = -x + 2$.

For $f(x,y) = y^2 - x^3 + x$, consider $(0,0)$. The line (at, ct) has $f(at, ct) = -a^3t^3 + c^2t^2 + at$, so it is tangent at $(0,0)$ if and only if $a = 0$, so the tangent line is $x = 0$.

For $f(x,y) = xy$, look again at $(0,0)$. The line (at, ct) has $f(at, ct) = act^2$, so we see that actually *every line* through the origin is tangent at $(0,0)$.

Finally, for the point $(0,0)$ and $f(x,y) = y^2 - x^3$, the line (at, ct) has $f(at, ct) = -a^3t^3 + c^2t^2$, so again every line is tangent at $(0,0)$.

Thus, the idea, which we shall make into a precise definition shortly, is that a point is singular if it has too many tangents. We first generalize to algebraic sets defined by more than one equation. The idea is that since $Z = \cap_{f \in I(Z)} Z(f)$, the same relation should hold on the set of tangent lines.

DEFINITION 4.1.4. Fix $Z \subseteq \mathbb{A}^n_k$ an affine algebraic set, and $P = (b_1, \dots, b_n) \in Z$. Then a parametric line $(x_1(t), \dots, x_n(t)) = (a_1t + b_1, \dots, a_nt + b_n)$ is **tangent** to Z at P if $f(a_1t + b_1, \dots, a_nt + b_n) \in k[t]$ vanishes to order at least 2 at $t = 0$ for all $f \in I(Z)$. In addition, we say a vector $(a_1, \dots, a_n) \in k^n$ is a **tangent vector** to Z at P if the line $(a_1t + b_1, \dots, a_nt + b_n)$ is tangent to Z at P.

The advantage of considering tangent vectors is that it turns out they always form a subspace of k^n. More precisely, we have the following.

EXERCISE 4.1.5. Let $X \subseteq \mathbb{A}^n_k$ be an affine algebraic set, with $f_1, \dots, f_m$ generating $I(X)$. Then given a point $P \in X$, and a vector $v \in k^n$, the following are equivalent:

(a) v is a tangent vector to Z at P;

(b) we have
$$(\partial f_i/\partial x_1(P), \dots, \partial f_i/\partial x_n(P)) \cdot v = 0$$
for $i = 1, \dots, m$;

(c) we have
$$(\partial f/\partial x_1(P), \dots, \partial f/\partial x_n(P)) \cdot v = 0$$
for all $f \in I(X)$.

In the above, the derivatives are purely formal, using the usual calculus rules for derivatives of polynomials, which make sense over any ring.

Since the conditions (b) and (c) of Exercise 4.1.5 are visibly linear, we can make the following definition.

DEFINITION 4.1.6. Given $Z \subseteq \mathbb{A}^n_k$ an affine algebraic set, and $P = (b_1, \dots, b_n) \in Z$, the **tangent space** of Z at P, denoted $T_P(Z)$, is the subspace of k^n consisting of tangent vectors to Z at P.

WARNING 4.1.7. Although the tangent space $T_P(Z)$ may be computed on a generating set for $I(Z)$, it is not true that if $Z = Z(I)$, then $T_P(Z)$ can be computed on a generating set for Z. The reason is that the definition of tangent space is quite sensitive to taking radicals. For instance, the origin in $\mathbb{A}^1_k$ has tangent space equal to (0), but it can also be written as $Z(x^2)$. If we tried to use x^2 to compute the tangent space, we would think that the tangent space is all of k.

We will see that the dimension of $T_P(Z)$ is always at least as large as the largest dimension of a component of Z containing P, and it will turn out that a point P of Z is nonsingular if and only if equality holds. However, we do not make this the definition, because it appears to depend on an imbedding in affine space. We will instead develop a definition which is visibly intrinsic.

4.2. Zariski cotangent spaces

We now describe a notion of (non)singularity which is intrinsic, and which also generalizes well. An important preliminary definition is the following.

Definition 4.2.1. Let R be a Noetherian local ring with maximal ideal $\mathfrak{m}$, and $k = R/\mathfrak{m}$. The **Zariski cotangent space** of R is defined to be $\mathfrak{m}_P/\mathfrak{m}_P^2$, considered as a k-vector space.

If X is an affine algebraic set, and $P \in X$, the **Zariski cotangent space** of X at P, denoted $T_P^*(X)$, is the Zariski cotangent space of the local ring $\mathscr{O}_{P,X}$.

If Noetherian local rings are too abstract, the prototype to have in mind are our local rings $\mathscr{O}_{P,X}$.

Note that k has a natural well-defined multiplication on $\mathfrak{m}/\mathfrak{m}^2$, so the first definition makes sense.

Why is it natural to think of $T_P^*(X)$ as the cotangent space? Intuitively, if we have a tangent vector v, and a function f on X, we can take the directional derivative of f in the direction of v, and obtain an element of k. This isn't affected if we subtract a constant from f, so we might as well assume that $f \in \mathfrak{m}_P$. On the other hand, if $f \in \mathfrak{m}_P^2$, then any directional derivative at P is always equal to 0, so we mod out by $\mathfrak{m}_P^2$ and arrive at the definition of $T_P^*(X)$.

Now, if R is a Noetherian local ring with maximal ideal $\mathfrak{m}$, then $\mathfrak{m}$ is finitely generated, and any set of generators for $\mathfrak{m}$ will also span $\mathfrak{m}/\mathfrak{m}^2$ as a vector space over $k := R/\mathfrak{m}$. Thus, we see that $\dim_k \mathfrak{m}/\mathfrak{m}^2$ is finite. On the other hand, Nakayama's lemma (see Corollary 4.8 of [**Eis95**]) implies that if we have $f_1, \dots, f_n \in \mathfrak{m}$ which span $\mathfrak{m}/\mathfrak{m}^2$, then $f_1, \dots, f_n$ also generate $\mathfrak{m}$. It then follows from Krull's principal ideal theorem (Theorem 2.4.15) that the height of $\mathfrak{m}$ is at most n, but in the local case, the height of $\mathfrak{m}$ is the same as $\dim R$, so we conclude:

Corollary 4.2.2. *If R is a Noetherian local ring, with maximal ideal $\mathfrak{m}$, and $k = R/\mathfrak{m}$, then*

$$\dim R \leqslant \dim_k \mathfrak{m}/\mathfrak{m}^2.$$

This leads us to the following definition:

Definition 4.2.3. A Noetherian local ring R with maximal ideal $\mathfrak{m}$ is **regular** if $\dim R = \dim_k \mathfrak{m}/\mathfrak{m}^2$, where $k = R/\mathfrak{m}$.

Definition 4.2.4. If X is an affine algebraic set, and $P \in X$, then X is **nonsingular** at P if $\mathscr{O}_{P,X}$ is regular. Otherwise, P is a **singularity** of X. We say X is **nonsingular** if it is nonsingular at P for all $P \in X$.

Remark 4.2.5. It is sometimes convenient to note that $\mathfrak{m}_P/\mathfrak{m}_P^2$ is the same whether we consider $\mathfrak{m}_P$ in the local ring $\mathscr{O}_{P,X}$, or in the affine coordinate ring $A(X)$. However, in the full coordinate ring, if $f_1, \dots, f_n$ span $\mathfrak{m}_P/\mathfrak{m}_P^2$, this does not necessarily imply that they generate $\mathfrak{m}_P$. The converse still holds, though.

Example 4.2.6. $\mathbb{A}_k^n$ is nonsingular. Indeed, if $P = (a_1, \dots, a_n)$ is any point, we already know that $\mathfrak{m}_P = (x_1 - a_1, \dots, x_n - a_n)$, so is generated by $n = \dim \mathbb{A}_k^n$ elements.

Note that in the example, what is happening is that each point P of $\mathbb{A}_k^n$ can be defined by n polynomials $f_1, \dots, f_n$ not only in the weak sense that $P = Z(f_1, \dots, f_n)$, but in the strong sense that $I(P) = (f_1, \dots, f_n)$. More generally, the foregoing algebra is saying that if we have $P \in X$ an affine algebraic set of dimension n near P, and $f_1, \dots, f_m \in \mathfrak{m}_P$ span $T_P^*(X)$, then in a neighborhood of P, we have

$$Z(f_1) \cap \cdots \cap Z(f_m) = \{P\},$$

and since each f_i can reduce the dimension by at most 1, we conclude that $n \leqslant m$. Nonsingularity then corresponds to the dimension of $\mathfrak{m}_P/\mathfrak{m}_P^2$ being equal to n, the minimal possible value, or equivalently, being able to find $f_1, \ldots, f_n \in A(X)$ whose zero sets cut out P in the strong sense that $(f_1, \ldots, f_n) = \mathfrak{m}_P$ (in a neighborhood of P). Thus, the f_i play a role like coordinate functions, and intuitively, the fact that they form a basis for $\mathfrak{m}_P/\mathfrak{m}_P^2$ means that their zero sets meet transversely. This motivates the following definition.

Definition 4.2.7. If X is an affine algebraic set, and $P \in X$ is a nonsingular point at which X has dimension d, a **system of local coordinates** for X at P is a d-tuple $(t_1, \ldots, t_d)$ of elements of the maximal ideal $\mathfrak{m}$ of $\mathscr{O}_{P,X}$ which induces a basis for the Zariski cotangent space $T_P^*(X)$.

In the case that $d = 1$ and (t) is a system of local coordinates, we will simply say that t is a **local coordinate**.

Example 4.2.8. Let's take another look at the curve Y given by $y^2 = x^3$. At the point $P = (1, 1)$ we have $\mathfrak{m}_P = (x - 1, y - 1)$. We can write

$$(x - 1)(x^2 + x + 1) = x^3 - 1 = y^2 - 1 = (y - 1)(y + 1),$$

and in the local ring, if $\operatorname{char} k \neq 2$, then $y + 1$ is invertible, so $y - 1$ is a multiple of $x - 1$, and $\mathfrak{m}_P$ is actually generated by $x - 1$. Thus, we see that $\dim \mathfrak{m}_P/\mathfrak{m}_P^2$ must be 1, and P is a nonsingular point, as we expect. If $\operatorname{char} k \neq 3$, then $x^2 + x + 1$ is invertible in $\mathscr{O}_{P,Y}$, so again we get that P is a nonsingular point, concluding that P is a nonsingular point in all characteristics.

On the other hand, if $P = (0, 0)$, the maximal ideal is (x, y). Then $\mathfrak{m}_P^2$ is generated by (x^2, xy, y^2), and in particular contains $y^2 - x^3$. This means that computing $\mathfrak{m}_P/\mathfrak{m}_P^2$ is the same in $A(Y)$ as in A_2. So $\mathfrak{m}_P/\mathfrak{m}_P^2$ is 2-dimensional, generated by x and y. Thus, P is a singularity of Y, as we would hope.

Remark 4.2.9. An immediate consequence of Corollary 4.2.2 is that if R is a Noetherian local ring, then $\dim R$ is finite. This is not otherwise obvious: certainly a Noetherian local ring can have infinite descending chains of ideals. Moreover, this statement does not hold for Noetherian rings in general, because although any given chain of prime ideals must be contained in a maximal chain of finite length (this follows from the local case), one can have incomparable chains of prime ideals whose lengths grow without bound. See Exercise 9.6 of [**Eis95**].

4.3. The Jacobian criterion

While we were able to investigate some examples, the definition of nonsingular point is often not very helpful in practice. To find a more tractable alternative, we return to the notion of tangent spaces, and relate nonsingularity to tangent spaces as follows.

Proposition 4.3.1. *Given an affine algebraic set $X \subseteq \mathbb{A}_k^n$, and $P \in X$, then $T_P^*(X)$ is the dual space of $T_P(X)$, via the pairing induced by*

$$\langle f, v\rangle = (\partial f/\partial x_1(P), \ldots, \partial f/\partial x_n(P)) \cdot v.$$

Proof. We first observe that the proposed pairing is well defined, which is precisely what Exercise 4.1.5 says, since if $\tilde{f}_1, \tilde{f}_2 \in A_n$ are two representatives of f, we have $\tilde{f}_1 - \tilde{f}_2 \in I(X)$.

Next, if we have $v \in T_P(X)$ such that for all $f \in \mathfrak{m}_P$, we see that $\langle f, v\rangle = 0$, then it is immediate that $v = 0$: if $P = (a_1, \ldots, a_n)$, then in particular we have $\langle x_i - a_i, v\rangle = 0$ for each i, but we see from the definition that $\langle x_i - a_i, v\rangle$ is simply the ith coefficient of v.

It remains to show that if we have $f \in \mathfrak{m}_P$ such that $\langle f, v\rangle = 0$ for all $v \in T_P(X)$, then $f \in \mathfrak{m}_P^2$ in $A(X)$. Letting $\tilde{\mathfrak{m}}_P$ denote the maximal ideal corresponding to P in the polynomial ring A_n, this is the same as showing that $f \in (\tilde{\mathfrak{m}}_P^2, I(X))$ in A_n. But we observe that $T_P(X)$ is the orthogonal complement to vectors of the form $(\partial g/\partial x_1(P), \ldots, \partial g/\partial x_n(P))$ for $g \in I(X)$, and by hypothesis,

$(\partial f/\partial x_1(P), \dots, \partial f/\partial x_n(P))$ is in the orthogonal complement to $T_P(X)$, so we conclude that there exists some $g \in I(X)$ such that

$$(\partial f/\partial x_1(P), \dots, \partial f/\partial x_n(P)) = (\partial g/\partial x_1(P), \dots, \partial g/\partial x_n(P)).$$

Then $f - g$ has all partial derivatives vanishing at P, so it is clear that it is in $\tilde{\mathfrak{m}}_P^2$, since if we expand in terms of the $x_i - a_i$, any nonzero linear term would create a nonzero partial derivative. □

This leads to a different interpretation of nonsingularity which is useful for both computational and theoretical purposes.

Corollary 4.3.2 (Jacobian criterion). *Suppose $X \subseteq \mathbb{A}_k^n$ is an affine algebraic set, and $f_1, \dots, f_m \in I(X)$. Given $P \in X$, if $\dim_P X = d$, then the $n \times m$ matrix*

$$J(f_1, \dots, f_m)(P) := ((\partial f_i/\partial x_j)(P))_{i,j}$$

has rank at most $n - d$. If equality holds, then P is a nonsingular point of X, and the converse holds if the f_i generate $I(X)$.

In particular, if $X = Z(f_1, \dots, f_m)$, and $J(f_1, \dots, f_m)(P)$ has rank m, then necessarily P is a nonsingular point of X and every component of X passing through P has dimension $n - m$.

In fact, we will see in Corollary 4.3.10 below that if X is nonsingular at P, then X has only one irreducible component containing P, so the phrasing "every component of X passing through P" above is a bit misleading.

Proof. First, Exercise 3.1.15 gives us that $\dim \mathscr{O}_{P,X} = d$. Let r be the rank of $J(f_1, \dots, f_m)(P)$. If we extend the f_i to a set $f_1, \dots, f_{m'}$ generating $I(X)$, and let r' be the rank of $J(f_1, \dots, f_{m'})(P)$, then Exercise 4.1.5 says that $T_P(X)$ is the orthogonal complement in k^n of the rowspace of $J(f_1, \dots, f_{m'})(P)$, so we have

$$\dim_k T_P(X) = n - r' \leqslant n - r.$$

By Proposition 4.3.1, we then have $\dim_k T_P^*(X) = n - r' \leqslant n - r$ also, so the first statements follow from the fact (Corollary 4.2.2) that $\dim_k T_P^*(X) \geqslant d$, and the definition of nonsingularity.

For the last statement, we already know from the Krull principal ideal theorem (Theorem 2.4.15) that every component of X has dimension at least $n - m$, but the hypotheses then imply that if Z is any component of X containing P, then

$$n - m = \dim_k T_P(X) \geqslant \dim Z \geqslant n - m,$$

so we must have equality and the statement follows. □

Example 4.3.3. Suppose $H = Z(f) \subseteq \mathbb{A}_k^n$ is a hypersurface, with f irreducible (so that $I(H) = (f)$). Then $P \in H$ is singular if and only if $(\partial f/\partial x_i)(P) = 0$ for each $i = 1, \dots, n$.

The power of the Jacobian criterion is that it allows us to analyze the entire set of singular points at once, instead of just looking at one point at a time.

Example 4.3.4. Returning to the plane curve with $f = y^2 - x^3$, we have $\partial f/\partial x = -3x^2$, and $\partial f/\partial y = 2y$. Certainly, we have a singular point at $(0,0)$, but if k doesn't have characteristic 3, setting $3x^2 = 0$ means $x = 0$ and the only point on the curve with $x = 0$ is $(0,0)$, so this is the only singularity. On the other hand, if k doesn't have characteristic 2, setting $2y = 0$ and $y^2 - x^3 = 0$ again implies the only singularity is $(0,0)$, so we conclude that $(0,0)$ is the only singularity independent of characteristic.

An important consequence of the Jacobian condition is the following:

Corollary 4.3.5. *Let $X \subseteq \mathbb{A}_k^n$ be an affine variety. Then the set of singular points of X form a proper (but possibly empty) closed subset of X.*

Proof. We first observe that we can view the Jacobian matrix $J(X)$ as a matrix with entries consisting of polynomials, and then $J(X)(P)$ is obtained simply by evaluating these polynomials at P. Hence, for any r, the set of $P \in \mathbb{A}^n_k$ such that $J(X)(P)$ has rank less than or equal to r defines a Zariski closed subset of $\mathbb{A}^n_k$, as it is described by the vanishing of $(r+1)\times(r+1)$ minors, which are polynomials. Now, if $\dim X = d$, the assertion that the set of singular points is closed is immediate: the singular points are simply the intersection of X with the subset of $\mathbb{A}^n_k$ on which $J(X)$ has rank less than or equal to $n-d-1$.

It remains to see that X has an open subset on which it is nonsingular. Here we use the theorem that X is birational to a hypersurface $H \subseteq \mathbb{A}^{d+1}_k$; since birational varieties have isomorphic open subsets, it is enough to prove that H is nonsingular on a nonempty open subset. Suppose $H = Z(f)$ for some irreducible f. Then the singular set of H is the intersection

$$Z(f) \cap Z(\partial f/\partial x_1) \cap \cdots \cap Z(\partial f/\partial x_n).$$

This can only be equal to all of H if $Z(f) \subseteq Z(\partial f/\partial x_i)$ for all i. But f is irreducible and $\partial f/\partial x_i$ has strictly smaller degree if it nonzero, so the only way this can happen is if $\partial f/\partial x_i = 0$ for all i. If k has characteristic 0 this is impossible for a nonconstant polynomial. If k has characteristic $p > 0$, it can only happen if in fact f is a polynomial in $x_1^p, \dots, x_n^p$. But in this case, since k is algebraically closed, by taking pth roots of the coefficients of f we can write $f = g^p$ for some polynomial g. This contradicts the irreducibility of f. We thus conclude that H, and hence X, has a nonempty subset of nonsingular points. $\square$

The following exercise relates systems of local coordinates more explicitly to the Jacobian criterion.

Exercise 4.3.6. Let $X \subseteq \mathbb{A}^n_k$ be an affine variety of dimension d, given by an ideal $I \subseteq k[t_1, \dots, t_n]$. Given $P = (c_1, \dots, c_n) \in X$, the following are equivalent:

(1) X is nonsingular at P of dimension d, and $(t_{n-d+1} - c_{n-d+1}, \dots, t_n - c_n)$ induces a system of local coordinates for X at P;
(2) projection to the last d coordinates induces an isomorphism $T_P(X) \xrightarrow{\sim} T_{(c_{n-d+1},\dots,c_n)}(\mathbb{A}^d_k)$;
(3) there exist $f_1, \dots, f_{n-d} \in I$ such that the Jacobian matrix $\left(\frac{\partial f_i}{\partial t_j}\right)_{1\leqslant i,j\leqslant n-d}$ is invertible at P.

We will conclude with a couple of additional fundamental results on (non)singularity, which require the following commutative algebra result:

Theorem 4.3.7. *If R is a regular local ring, then R is an integral domain.*

See Corollary 10.14 of [**Eis95**].

We also have the following easier lemma:

Exercise 4.3.8. Let R be a ring, and $\mathfrak{p}_1, \dots, \mathfrak{p}_n, \mathfrak{q}_1, \dots, \mathfrak{q}_m$ prime ideals such that no $\mathfrak{p}_i$ is contained in any $\mathfrak{q}_j$. Then there exists $f \in R$ such that $f \in \mathfrak{p}_i$ for $i = 1, \dots, n$ but $f \notin \mathfrak{q}_j$ for $j = 1, \dots, m$.

Corollary 4.3.9. *If X is an affine algebraic set, then $\mathscr{O}_{P,X}$ is an integral domain if and only if P is contained in a unique irreducible component of X.*

Proof. If P is contained in a unique irreducible component of X, then then the complement of the other irreducible components is an open neighborhood U of P which is irreducible. It is then clear that $\mathscr{O}_{P,X}$ is an integral domain, since we can replace X by the closure of U without changing $\mathscr{O}_{P,X}$. Conversely, suppose that $\mathscr{O}_{P,X}$ is an integral domain, let Z be an irreducible component of X containing P, and Z' the union of the other irreducible components of X. Using Exercise 4.3.8, let $f_1 \in A(X)$ vanish on Z but not on any component of Z', and let $f_2 \in A(X)$ vanish on Z' but not on Z. Then $f_1 f_2 = 0$, so we must have either f_1 or f_2 equal to 0 in $\mathscr{O}_{P,X}$. But because $P \in Z$, we cannot have $f_2 = 0$ in $\mathscr{O}_{P,X}$. Thus $f_1 = 0$ in $\mathscr{O}_{P,X}$, and we conclude that no component of Z' can contain P, as desired. $\square$

For us, the main geometric consequence of Theorem 4.3.7 is the following, which now follows immediately from Corollary 4.3.9.

Corollary 4.3.10. *If X is an affine algebraic set, and $P \in X$ is a nonsingular point, then P is contained in a unique irreducible component of X.*

In particular, a connected nonsingular algebraic set is automatically a variety. This fits with intuition – a point where two components intersect should not be nonsingular – but it's not so obvious from either definition. We can use this to generalize Corollary 4.3.5.

Corollary 4.3.11. *If X is an affine algebraic set, then the set of singular points of X is a nowhere dense closed subset of X.*

Proof. Let $Z \subseteq X$ be the points which are contained in more than one irreducible component of X; then Z is closed. If $U = X \smallsetminus Z$, then U is open and dense in X, and is a disjoint union of components $U_1, \dots, U_n$ whose closures $Z_1, \dots, Z_n$ are the irreducible components of X. Because local rings aren't affected by restricting to open subsets, the nonsingular points of X which are contained in U are exactly the nonsingular points of each Z_i which are contained in U_i, so these form a dense open subset of U by Corollary 4.3.5. On the other hand, by Corollary 4.3.10, we see that none of the points of Z are nonsingular points of X, so we conclude that the nonsingular points of X are a dense open subset of U, and hence of X. $\square$

Finally, we address an additional consequence of the results we have developed so far. Recall that if an affine variety X has codimension c in $\mathbb{A}^n_k$, it need not be possible to define X using only c polynomials. We express this by saying that not every variety is a *complete intersection*. In fact, it is not difficult to check that even if we relax the condition, and ask only that every point of X has a neighborhood on which X can be defined by c polynomials, that this is still not always possible. This says that X need not be a *local complete intersection*. However, we see that nonsingular varieties are always local complete intersections.

Corollary 4.3.12. *If $P \in X$ is a nonsingular point, then X is a local complete intersection at P, in the following sense: if $\dim_P X = d$, we have $X \subseteq \mathbb{A}^n_k$, and $f_1, \dots, f_m \in A_n$ generating $I(X)$, choose $i_1, \dots, i_{n-d}$ so that the corresponding rows of $J(X)(P)$ are linearly independent; then there exists an open neighborhood U of P in $\mathbb{A}^n_k$ such that*

$$Z(f_{i_1}, \dots, f_{i_{n-d}}) \cap U = X \cap U.$$

Proof. Set $Z = Z(f_{i_1}, \dots, f_{i_{n-d}})$. Obviously, we have $X \subseteq Z$. On the other hand, every irreducible component of Z must have dimension at least d, and applying the Jacobian criterion to Z, we see that it is nonsingular at P, with dimension d in a neighborhood of P. But then by Corollary 4.3.10 Z has a unique irreducible component Z' containing P, which has dimension equal to that of X. It then follows that X must also contain Z', and choosing U to be the complement of any other components of X or Z, we have $X \cap U = Z' \cap U = Z \cap U$, as desired. $\square$

Remark 4.3.13. When one studies this sort of question, there are two different strengths with which one can formulate the question. The weaker version is to look for $f_1, \dots, f_c$ such that $X = Z(f_1, \dots, f_c)$. The stronger one is to actually ask that $f_1, \dots, f_c$ generate $I(X)$. The above argument is phrased in the context of the weaker version for the sake of conceptual simplicity, but the stronger statement also follows by arguments which are more algebraic. See Corollary 4.A.2 below.

REMARK 4.3.14. Our presentation is ahistorical in that the notion of nonsingularity in terms of partial derivatives far preceded the theory of regular local rings (developed by Krull) and its connection to nonsingularity (developed by Zariski) in the 1930's and 1940's.

EXERCISE 4.3.15. Show that if $f(x, y) \in k[x, y]$ is of the form $y^2 - g(x)$ for a nonconstant polynomial $g(x) \in k[x]$, so that $f(x, y)$ is irreducible by Exercise 2.2.20, then $Z(f(x, y))$ is nonsingular if and only if $g(x)$ has no multiple roots.

EXERCISE 4.3.16. Show that if X and Y are nonsingular affine varieties, then $X \times Y$ is also nonsingular.

EXERCISE 4.3.17. Show that if Y is a nonsingular affine variety of dimension d, and $Z \subseteq Y$ is a nonsingular closed subvariety of codimension c, then for every point $P \in Z$ there exist $f_1, \dots, f_c \in A(Y)$ and an open neighborhood U of P in Y such that

$$Z \cap U = Z(f_1, \dots, f_c) \cap U \subseteq Y.$$

EXERCISE 4.3.18. Let $Y \subseteq \mathbb{A}^n_k$ be a nonsingular affine variety, and Z_1, Z_2 closed subvarieties of Y. Suppose Z is an irreducible component of $Z_1 \cap Z_2$. Show that

$$\operatorname{codim}_Y Z \leqslant \operatorname{codim}_Y Z_1 + \operatorname{codim}_Y Z_2.$$

Hint: express $Z_1 \cap Z_2$ as a different intersection inside $Y \times Y \subseteq \mathbb{A}^{2n}_k$.

4.A. Local generation of ideals

We explain the stronger, ideal-theoretic formulation of local complete intersections at nonsingular points, and give an application to generation of ideals.

LEMMA 4.A.1. *Let $X \subseteq \mathbb{A}^n_k$ be an algebraic set, of dimension d at a nonsingular point $P \in X$. If $f_1, \dots, f_{n-d} \in I(X)$ has $J(f_1, \dots, f_{n-d})(P)$ of maximal rank $n - d$, then:*

(i) *$I(X)\mathscr{O}_{P,\mathbb{A}^n_k} = (f_1, \dots, f_{n-d})\mathscr{O}_{P,\mathbb{A}^n_k}$;*
(ii) *there exists $f \in A(\mathbb{A}^n_k) \smallsetminus \mathfrak{m}_P$ such that $I(X)A(\mathbb{A}^n_k)_f = (f_1, \dots, f_{n-d})A(\mathbb{A}^n_k)_f$.*

PROOF. Let $\mathfrak{m}_P \subseteq \mathscr{O}_{P,X}$ and $\mathfrak{n}_P \subseteq \mathscr{O}_{P,\mathbb{A}^n_k}$ be the respective maximal ideals, and $I' = I(X)\mathscr{O}_{P,\mathbb{A}^n_k}$, so that $\mathfrak{m}_P = \mathfrak{n}_P/I'$. Then we have an exact sequence of k-vector spaces

$$0 \to I'/\mathfrak{n}_P I' \to \mathfrak{n}_P/\mathfrak{n}_P^2 \to \mathfrak{m}_P/\mathfrak{m}_P^2 \to 0.$$

We know that $\dim_k \mathfrak{n}_P/\mathfrak{n}_P^2 = n$ and $\dim_k \mathfrak{m}_P/\mathfrak{m}_P^2 = d$, so we conclude that $\dim_k I'/\mathfrak{n}_P I' = n - d$. But applying Proposition 4.3.1 to the case of $\mathbb{A}^n_k$, our hypothesis on $\operatorname{rk} J(f_1, \dots, f_{n-d})(P)$ implies that (the images of) $f_1, \dots, f_{n-d}$ are linearly independent in $\mathfrak{n}_P/\mathfrak{n}_P^2$, and hence must give a basis of $I'/\mathfrak{n}_P I'$. Then (i) follows by Nakayama's lemma.

Now, (i) means that if $(g_1, \dots, g_\ell)$ is a generating set for $I(X)$, then for each $j = 1, \dots, \ell$, we have an expression

$$g_\ell/1 = \sum_{i=1}^{m} a_{i,j}/b_{i,j} f_i$$

in $\mathscr{O}_{P,\mathbb{A}^n_k}$, where $a_{i,j} \in A(\mathbb{A}^n_k)$, and $b_{i,j} \in A(\mathbb{A}^n_k) \smallsetminus \mathfrak{n}_P$. Setting f to be the product of the $f_{i,j}$ will then yield (ii). $\square$

We rephrase the second part of the lemma more geometrically as follows.

COROLLARY 4.A.2. *If $P \in X \subseteq \mathbb{A}^n_k$ is a nonsingular point of an algebraic set, with $\dim_P X = d$, and if we choose $f_1, \dots, f_{n-d} \in I(X)$ so that $J(f_1, \dots, f_{n-d})(P)$ has rank $n - d$, then there exists $f \in A(\mathbb{A}^n_k)$ such that $f(P) \neq 0$, and*

$$I(X \smallsetminus Z(f)) = (f_1, \dots, f_{n-d}).$$

More precisely, let $Z = Z(x_{n+1}f - 1) \subseteq \mathbb{A}_k^{n+1}$*, so that the projection morphism* $\mathbb{A}_k^{n+1} \to \mathbb{A}_k^n$ *induces an isomorphism*

$$\varphi : Z \xrightarrow{\sim} \mathbb{A}_k^n \smallsetminus Z(f).$$

Then $I(\varphi^{-1}(X \smallsetminus Z(f))) = (f_1 \circ \varphi, \ldots, f_{n-d} \circ \varphi, x_{n+1}f - 1)$.

PROOF. This is essentially a rephrasing of Lemma 4.A.1 (ii), recalling that the map $A(\mathbb{A}_k^n) \to A(Z)$ is canonically identified with the map $A(\mathbb{A}_k^n) \to A(\mathbb{A}_k^n)_f$. □

We can apply these ideas as follows.

COROLLARY 4.A.3. *Let* $X \subseteq \mathbb{A}_k^n$ *be a variety of dimension* d*, and* $f_1, \ldots, f_m \in I(X)$ *such that* $Z(f_1, \ldots, f_m)$ *is connected, and* $\operatorname{rk} J(f_1, \ldots, f_m)(P) = n - d$ *for all* $P \in X$*. Then* $I(X) = (f_1, \ldots, f_m)$.

Note that the connectedness hypothesis is in particular satisfied if we are given that $Y = Z(f_1, \ldots, f_m)$, so that Corollary 4.A.3 is one approach for showing that polynomial equations which cut out a variety in fact generate the ideal of the variety.

PROOF. At each $P \in X$, we can choose a subset of $n - d$ of the f_i corresponding to linearly independent rows of the Jacobian matrix, and then by Lemma 4.A.1 (i) we see that $I(X)\mathscr{O}_{P,\mathbb{A}_k^n} = (f_1, \ldots, f_m)\mathscr{O}_{P,\mathbb{A}_k^n}$ for all $P \in X$.

Next, we claim that $X = Z(f_1, \ldots, f_m)$. Since $X \subseteq Z(f_1, \ldots, f_m)$ by hypothesis, we have by the Jacobian criterion that for all $P \in X$, the zero set $Z(f_1, \ldots, f_m)$ is nonsingular of dimension d at P, so $Z(f_1, \ldots, f_m)$ must have X as one of its irreducible components, and no other component can intersect X by Corollary 4.3.10. But then our connectedness hypothesis implies $Z(f_1, \ldots, f_m)$ cannot have any other irreducible components, proving the claim.

Now, if $P \notin X$, then $I(X) \not\subseteq \mathfrak{m}_P$, so $I(X)$ contains a unit in $\mathscr{O}_{P,\mathbb{A}_k^n}$, and $I(X)\mathscr{O}_{P,\mathbb{A}_k^n} = \mathscr{O}_{P,\mathbb{A}_k^n}$. But the same holds for $(f_1, \ldots, f_m)$, so we conclude that we have $I(X)\mathscr{O}_{P,\mathbb{A}_k^n} = (f_1, \ldots, f_m)\mathscr{O}_{P,\mathbb{A}_k^n}$ for all $P \in \mathbb{A}_k^n$. The following algebra proposition then completes our argument. □

PROPOSITION 4.A.4. *Let* R *be a ring, and* $I \subseteq I' \subseteq R$ *ideals. If* $IR_{\mathfrak{m}} = I'R_{\mathfrak{m}}$ *for every maximal ideal of* R*, then* $I = I'$.

PROOF. Suppose we have $x \in I'$. Let $J = \{y \in R : yx \in I\}$. We observe that J is an ideal, and wish to show that $J = R$. But given a maximal ideal $\mathfrak{m}$ of R, we have $IR_{\mathfrak{m}} = I'R_{\mathfrak{m}}$, which means we have $x/1 = r/s$ for some $r \in I$, and $s \notin \mathfrak{m}$, or equivalently, there exists $z \notin \mathfrak{m}$ such that $xsz = rz$. We then see that $sz \in J$, and $sz \notin \mathfrak{m}$, so $J \not\subseteq \mathfrak{m}$. Since this holds for all $\mathfrak{m}$, we conclude that $J = R$, as desired. □

CHAPTER 5

Abstract varieties via atlases

Having discussed affine and quasiaffine varieties, we next describe how abstract algebraic varieties over an algebraically closed field may be defined rigorously via an atlas definition analogous to the usual definition of differential manifolds. The restriction to algebraically closed fields allows us to use the naive notions of the underlying spaces and their morphisms; we thus fix throughout an algebraically closed field k.

5.1. Prevarieties

In the definition of manifolds, one imposes a condition that every point have a neighborhood which is isomorphic (in some appropriate sense) to an open subset of $\mathbb{R}^n$. This is not possible for varieties, for two reasons. The first is that we do not want to restrict ourselves to smooth varieties, so even for complex varieties we will not necessarily obtain topological manifolds. However, even if we wished to restrict to smooth varieties, we could still not ask for our varieties to be locally isomorphic to an open subset of affine space, because algebraic maps are fundamentally much more rigid than differentiable maps, and it is simply not the case that a smooth variety has an open cover by varieties which can be thought of as open subvarieties of affine space. We therefore allow our varieties instead to be covered by open subsets which are isomorphic to affine varieties.

Asking for a variety to have an open cover by affine varieties may at first be counterintuitive – in analogy with manifold theory, why not allow the open subsets to be isomorphic to open subsets of affine varieties? However, we have just seen that every open subset of an affine variety has an open cover by affine varieties, so in fact this would not be any different, and we conserve words by imposing that our open cover consist of affine varieties.

To summarize, we will construct general abstract varieties by gluing together affine varieties along open subsets, with the restriction that the gluing maps must be algebraic. We have:

Definition 5.1.1. A **prevariety** X over k is an irreducible topological space, together with an open cover $U_1, \dots, U_m$, and a collection of homeomorphisms $\varphi_i : X_i \xrightarrow{\sim} U_i$, where each $X_i \subseteq \mathbb{A}^{n_i}$ is an affine variety equipped with the Zariski topology, and we require that every **transition map**

$$\varphi_{i,j} : \varphi_i^{-1}(U_i \cap U_j) \overset{\varphi_j^{-1}\circ\varphi_i}{\xrightarrow{\sim}} \varphi_j^{-1}(U_i \cap U_j)$$

is a morphism (of quasiaffine varieties). We say that each map $\varphi_i : X_i \to U_i \subseteq X$ is a **chart**, and the collection of charts is an **atlas**.

Remarks 5.1.2. Since $\varphi_{i,j}^{-1} = \varphi_{j,i}$, the transition maps are necessarily isomorphisms. One often thinks of a prevariety as being obtained from the collection of affine varieties X_i by gluing together open subsets along the isomorphisms given by the transition maps.

We have not yet defined varieties because we haven't yet imposed the condition analogous to the Hausdorff condition for a manifold. We will revisit this shortly.

One can vary the definition a bit by defining a notion of equivalence of atlases and speaking of a prevariety as a set with an equivalence class of atlases, or alternatively, by requiring an atlas to be maximal. Either of these options removes the "dependence on choice" of the atlas, but at this point it is not clear whether it would be any less technical to simply do what modern algebraic geometers do, which is to work with sheaves.

EXAMPLE 5.1.3. Any affine variety "is" a prevariety, with an atlas consisting of a single chart.

EXAMPLE 5.1.4. Suppose we have $U_1 = U_2 = \mathbb{A}^1_k$, and set $U = \mathbb{A}^1_k \smallsetminus (0)$. We consider two different possibilities for gluing U_1 to U_2 along U to obtain a prevariety.

The first is to let X be the union of U_1 and U_2 glued along U, where we identify $U \subseteq U_1$ and $U \subseteq U_2$ simply by the identity map. In this case, X is almost the same as $\mathbb{A}^1_k$, except that now it has two copies of the origin instead of one. In the usual real or complex topology, this satisfies the conditions to be a manifold except that it is not Hausdorff.

On the other hand, we could identify the points of $U \subseteq U_1$ with $U \subseteq U_2$ via the inversion map $t \mapsto 1/t$. As we will see later, this is one way of describing the projective line $\mathbb{P}^1_k$. In the usual real or complex topology, this will give a manifold, and will in fact be compact. We can picture that by adding in U_2 we have compactified U_1 – because our transition map is $t \mapsto 1/t$, the origin of U_2 becomes the "point at infinity" of U_1.

In the following exercise, we explore when we can in fact glue together affine varieties to make a prevariety.

EXERCISE 5.1.5. Suppose that $X_1, \dots, X_n$ is a collection of affine varieties, and for each i, j, we have $U_{i,j} \subseteq X_i$ an open subset, and $\varphi_{i,j} : U_{i,j} \to U_{j,i}$ an isomorphism of quasiaffine varieties. Suppose further that the $\varphi_{i,j}$ satisfy the **cocycle condition**: for each i, j, ℓ,

$$\varphi_{j,\ell} \circ \left(\varphi_{i,j}|_{\varphi_{i,j}^{-1}(U_{j,\ell})}\right) = \varphi_{i,\ell}|_{\varphi_{i,j}^{-1}(U_{j,\ell})}.$$

(a) Show that there is a topological space X with an open cover $U_1, \dots, U_n$ and homeomorphisms $\varphi_i : X_i \to U_i$ for $i = 1, \dots, n$ such that the induced transition maps are the $\varphi_{i,j}$ (in particular, X has the structure of a prevariety).

(b) Show that the construction of (a) is unique in the sense that if $X', U'_1, \dots, U'_n, \varphi'_1, \dots, \varphi'_n$ satisfies the same conditions, then there is a unique homeomorphism $\varphi : X \to X'$ such that $\varphi(U_i) = U'_i$ and $\varphi \circ \varphi_i = \varphi'_i$ for each i.

(c) Finally, suppose that, given $X_1, \dots, X_n$ as above, for each i we have fixed isomorphisms between $K(X_i)$ and a given field K (over k). Given also $U_{i,j}$ as above, suppose that the isomorphisms $\varphi_{i,j}$ are induced by our isomorphisms $K(X_j) \cong K \cong K(X_i)$. Show that the $\varphi_{i,j}$ automatically satisfy the cocycle condition.

The first order of business is to understand quasiaffine varieties as special cases of prevarieties. We make the following definition:

DEFINITION 5.1.6. If X is a quasiaffine variety, and we have an atlas $\{\varphi_i : X_i \to U_i\}$ on the underlying topological space of X, we say the atlas is **admissible** if each of the φ_i is an isomorphism of quasiaffine varieties.

We see immediately that the one-chart atlas of Example 5.1.3 is admissible. More generally, we will check:

PROPOSITION 5.1.7. *Suppose X is a quasiaffine variety and $\{\varphi_i : X_i \to U_i\}$ is a finite collection of isomorphisms, with the X_i being affine varieties, and the U_i forming an open cover of X (and each U_i considered as a quasiaffine variety). Then $\{\varphi_i : X_i \to U_i\}$ is an admissible atlas on X. In particular, every quasiaffine variety has an admissible atlas.*

PROOF. For the first statement, all we need to check is that the transition maps are morphisms, and in fact we see that they are isomorphisms because they are compositions of isomorphisms of quasiaffine varieties.

The second statement is immediate, since we know by Corollary 3.2.10 that X is covered by open subsets which are isomorphic to affine varieties. $\square$

Unless we state otherwise, we will from now on always assume that an atlas on a quasiaffine variety is admissible.

We next consider more generally what sort of atlases we will want to use on a (suitable) subset of a prevariety.

Proposition 5.1.8. *Let X be a prevariety with atlas $\{\varphi_i : X_i \to U_i\}_{i\in I}$, and let Z be an irreducible closed subset of an open subset of X. Then Z has an atlas $\{\psi_j : Z_j \to V_j\}_{j\in J}$ with the following property: for all $i \in I$ and $j \in J$, the induced map*

$$\psi_j^{-1}(U_i) \overset{\psi_j}{\to} V_j \cap U_i \hookrightarrow U_i \overset{\varphi_i^{-1}}{\to} X_i$$

gives an isomorphism of $\psi_j^{-1}(U_i) \subseteq Z_j$ onto its image in X_i.

Note that because ψ_j and φ_i are homeomorphisms by hypothesis, the image of $\psi_j^{-1}(U_i)$ in X_i is locally closed, so may be considered as a quasiaffine variety.

Proof. For each i, let $\{\psi_{i,j} : Z_{i,j} \to V_{i,j}\}_{j\in J_i}$ be an admissible atlas of the quasiaffine variety $\varphi_i^{-1}(Z) \subseteq X_i$. Then let $J = \coprod_i J_i$, and form $\{\psi_j : Z_j \to V_j\}_{j\in J}$ simply by taking the union of all of the $\varphi_i \circ \psi_{i,j'}$. We claim that this is an atlas with the desired property. It is evident that each $\psi_j = \varphi_i \circ \psi_{i,j'}$ is a homeomorphism onto an open subset of $Z \cap X_i$ and hence of Z, and by construction the union of the images covers Z.

First, given j_1 corresponding to i_1, j_1' and j_2 corresponding to i_2, j_2', the transition map $\psi_{j_2}^{-1} \circ \psi_{j_1}^{-1}$ is obtained as

$$\psi_{i_2,j_2'}^{-1} \circ \varphi_{i_2}^{-1} \circ \varphi_{i_1} \circ \psi_{i_1,j_1'},$$

and by hypothesis $\varphi_{i_2}^{-1} \circ \varphi_{i_1}$, $\psi_{i_2,j_2'}$, and $\psi_{i_1,j_1'}$ are isomorphisms, so we conclude that the transition map is likewise an isomorphism. Similarly, any $\varphi_i^{-1} \circ \psi_j$ is obtained as

$$\varphi_i^{-1} \circ \varphi_{i'} \circ \psi_{i',j'}$$

for some i', j', so is an isomorphism onto its image. $\square$

We thus define:

Definition 5.1.9. If X is a prevariety, a **subprevariety** of X is an irreducible closed subset of an open subset of X, with an atlas of the form given in Proposition 5.1.8.

Example 5.1.10. Although in general an induced atlas on an open subset will involve introducing more charts, this need not always be the case. For instance, if X is any prevariety with atlas $\{\varphi_i : X_i \to U_i\}$, and we consider the subprevariety of X corresponding to U_i, we see that we may take the one-element atlas on U_i consisting only of $\varphi_i : X_i \to U_i$.

5.2. Regular functions and morphisms

We next wish to define morphisms of prevarieties. Following the approach of [**Har77**] for quasiaffine varieties, we first define regular functions, and use these to define morphisms.

Definition 5.2.1. If X is a prevariety with a given atlas, and $U \subseteq X$ is open, a function $f : U \to k$ is **regular** on $U \subseteq X$ if for all i, the induced function

$$f \circ \varphi_i : \varphi_i^{-1}(U) \to k$$

is regular (in the sense defined for quasiaffine varieties).

As before, we denote by $\mathscr{O}(U)$ the ring of regular functions on $U \subseteq X$.

We incorporate X into our notation to reinforce that *a priori*, the notion of regularity depends on the choice of atlas on X.

We first wish to verify that this concept of regular function is completely compatible with the definition for quasiaffine varieties.

PROPOSITION 5.2.2. *If X is a quasiaffine variety, and $U \subseteq X$ open, then a function $f : U \to k$ is regular on X in the quasiaffine sense if and only if f is regular on X as a prevariety (always assumed to have an admissible atlas).*

PROOF. Let $\{\varphi_i : X_i \to U_i\}$ be the given (admissible) atlas on X. By definition of admissibility and the fact that regularity in the quasiaffine sense is a local condition, we have that quasiaffine regularity on U is equivalent to quasiaffine regularity on $U \cap U_i$ for all i, which is equivalent to quasiaffine regularity on $\varphi_i^{-1}(U)$ for all i, giving the desired statement. $\square$

It would also be good to know that our definition of subprevariety is in some sense independent of the choices involved in the induced atlas. We will show that the resulting regular functions are independent of these choices. Rather than showing this directly, we give a visibly independent characterization of regular functions, which is also useful in other contexts. We have:

PROPOSITION 5.2.3. *Let X be a prevariety, and $Z \subseteq X$ a subprevariety. Given $U \subseteq Z$ open and $f : U \to k$, we have f regular on $U \subseteq Z$ if and only if for all $P \in U$, there exists $V \subseteq X$ an open neighborhood of P, and a regular function $g : V \to k$ on $V \subseteq X$ such that $f|_{V \cap U} = g|_{V \cap U}$.*

COROLLARY 5.2.4. *If X is a prevariety, and $Z \subseteq X$ a subprevariety, then for any $U \subseteq Z$ open, the regularity of a function $f : U \to k$ is independent of the choice of atlas on Z as a subprevariety of X.*

We now move on to the definition of morphisms and isomorphisms. In particular, this will help us understand when two atlases should be considered equivalent.

We can define morphisms as usual:

DEFINITION 5.2.5. Given prevarieties X, Y with atlases given by $\{\varphi_i : X_i \overset{\sim}{\to} U_i\}_i$ and $\{\psi_j : Y_j \overset{\sim}{\to} V_j\}_j$, a **morphism** $\varphi : X \to Y$ is a continuous map such that for all $U \subseteq Y$ open, and all $f : U \to k$ regular, we have $f \circ \varphi : \varphi^{-1}(U) \to k$ is also regular.

It is immediate from Proposition 5.2.2 that if X, Y are quasiaffine varieties, morphisms $X \to Y$ in the above sense are the same as morphisms in the sense we had already defined. We also see from the definition (technically, using that composition of functions is associative) that compositions of morphisms are morphisms.

EXERCISE 5.2.6. Using the construction of the projective line described in Example 5.1.4, compute the regular functions which are defined on the whole projective line. Conclude that the projective line is not isomorphic to any quasiaffine variety.

EXAMPLE 5.2.7. If X is a prevariety and $\{\varphi_i : X_i \to U_i\}$ an atlas, then for each i, considering U_i as a prevariety, the chart map φ_i is an isomorphism. Indeed, we saw in Example 5.1.10 that U_i has a one-element atlas given by $\varphi_i : X_i \to U_i$, so by definition of regular functions on prevarieties, φ_i induces a correspondence between regular functions on (open subsets) of U_i and on X_i, and thus φ_i is an isomorphism.

We next see that the condition of being a morphism is a local one.

EXERCISE 5.2.8. If $\varphi : X \to Y$ is a morphism, and $U \subseteq X$ is an open subset considered as a prevariety, then $\varphi|_U$ is a morphism.

Conversely, if we have a map $\varphi : X \to Y$ and an open cover U_i of X such that $\varphi|_{U_i}$ is a morphism for each i, then φ is a morphism.

Morphisms are also local on the target.

EXERCISE 5.2.9. If $\varphi : X \to Y$ is a morphism, and $V \subseteq Y$ is an open subset of Y, if we consider V and $\varphi^{-1}(V)$ as prevarieties, then φ induces a morphism $\varphi^{-1}(V) \to V$.

Conversely, a continuous map $\varphi : X \to Y$ is a morphism if there is some open cover V_i of Y such that φ induces a morphism $\varphi^{-1}(V_i) \to V_i$ for each i.

A definition of morphism more analogous to a typical one using atlases for differentiable manifolds is then the following:

EXERCISE 5.2.10. With notation as in the above definition, a continous map $\varphi : X \to Y$ is a morphism if and only if for any i, j, the induced map

$$(\varphi_i)^{-1}(\varphi^{-1}(V_j)) \xrightarrow{\varphi_i} \varphi^{-1}(V_j) \cap U_i \xrightarrow{\varphi} V_j \xrightarrow{(\psi_j)^{-1}} Y_j$$

is a morphism of quasiaffine varieties.

Following are some basic properties of morphisms.

EXERCISE 5.2.11. Prove the following.

(a) If $Z \subseteq X$ is a subprevariety of a prevariety, the inclusion map of prevarieties is a morphism.
(b) If X and Y are prevarieties, Z a subprevariety of Y, and $\varphi : X \to Y$ any map with $\varphi(X) \subseteq Z$, then φ is a morphism if and only if the induced map $X \to Z$ is a morphism.

Suppose X is a prevariety, and $Y \subseteq \mathbb{A}^n$ an affine variety. Then we see that a function $X \to Y$ is equivalent to an n-tuple of functions $X \to k$, such that the induced map $X \to \mathbb{A}^n$ factors through Y. We can effectively use this correspondence to describe morphisms in terms of n-tuples of regular functions, generalizing our previous results for the quasiaffine case.

PROPOSITION 5.2.12. *Given a prevariety X and an affine variety $Y \subseteq \mathbb{A}^n$, morphisms $X \to Y$ are equivalent to n-tuples of regular functions on X such that the induced map $\varphi : X \to \mathbb{A}^n$ has image contained in Y.*

In particular, morphisms $X \to Y$ are in bijection with k-algebra homomorphisms $A(Y) \to \mathscr{O}(X)$.

PROOF. Certainly, if φ is a morphism, then pulling back the coordinate functions $x_1, \dots, x_n$ on $\mathbb{A}^n$ gives an n-tuple of regular functions on X, which describe φ. Conversely, suppose the pullbacks of the x_i are regular functions $f_i \in \mathscr{O}(X)$, so we have for any j that $f_i \circ \varphi_j$ is regular on X_j, using our standard atlas notation. But X_j is affine, so this means that the induced map $X_j \to Y$ is a morphism in the classical sense, and (using the one-chart atlas for Y) by Exercise 5.2.10 we conclude that φ is a morphism.

Now, to prove that morphisms $X \to Y$ are in bijection with k-algebra homomorphisms $A(Y) \to \mathscr{O}(X)$ it is enough to prove that an n-tuple of regular functions $f_1, \dots, f_n$ on X defines a function which maps X into Y if and only if $g(f_1, \dots, f_n) = 0$ for all $g \in I(Y)$, and this proceeds just as in the quasiaffine case, described in Corollary 3.2.11. $\square$

5.3. Abstract varieties

Note that a prevariety X is never Hausdorff (unless X consists of a single point), since it is irreducible by hypothesis. However, the analogue of the Hausdorff condition for a manifold is precisely what is missing from our definition. It turns out that the right definition involves the following fact from point-set topology:

EXERCISE 5.3.1. A topological space X is Hausdorff if and only if the image of the diagonal map $X \to X \times X$ is closed.

We will use the same definitions for varieties, but because the Zariski topology on a product of varieties is not the product topology, we will obtain a different and better-behaved notion.

Of course, we first need to define the product of prevarieties. The affine case was explored in Exercise 3.2.22, but we recall the definition.

Definition 5.3.2. Given $X \subseteq \mathbb{A}_k^n$ and $Y \subseteq \mathbb{A}_k^m$ affine varieties, let $X \times Y \subseteq \mathbb{A}_k^{n+m}$ be the product set via the natural identification of sets $\mathbb{A}_k^n \times \mathbb{A}_k^m = \mathbb{A}_k^{n+m}$.

We then generalize as follows.

Definition 5.3.3. Given prevarieties X, Y, we define the **product** $X \times Y$ of X with Y to be the product set $X \times Y$, equipped with the atlas

$$\varphi_i \times \varphi_j : X_i \times Y_j \xrightarrow{\sim} U_i \times V_j,$$

and the topology induced by the atlas.

Exercise 5.3.4. Show the following:

(a) The above definition gives a valid prevariety.
(b) If X and Y are affine, this definition is consistent with the one we already have (and used in the above) for products of affine varieties.
(c) If $Y \subseteq X$ is a subprevariety, then the topology on $Y \times Y \subseteq X \times X$ is the subset topology.
(d) The projection maps $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$ are morphisms.
(e) If Z is any prevariety, then a map $Z \to X \times Y$ is a morphism if and only if the induced maps $Z \to X, Z \to Y$ are morphisms, where the induced maps are obtained by composing with p_1 and p_2.

We then have that any prevariety X has a natural diagonal morphism:

Corollary 5.3.5. *Given a prevariety X, the diagonal map $\Delta : X \to X \times X$ is a morphism of prevarieties.*

Proof. This is immediate from Exercise 5.3.4 (d), since Δ is the unique morphism corresponding to the identity map on each factor. $\square$

Our analogy to the Hausdorff condition is then the following:

Definition 5.3.6. We say that a prevariety X is a **variety** if the image of the diagonal morphism is closed.

Example 5.3.7. Any affine variety is a variety. Indeed, if $X = Z(f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n))$ is an affine variety in $\mathbb{A}^n$, then

$$\Delta(X) = Z(f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n), x_1 - x_{n+1}, x_2 - x_{n+2}, \ldots, x_n - x_{2n}),$$

so is closed in $\mathbb{A}^{2n}$.

Example 5.3.8. An example of a prevariety which is not a variety is given by considering X to be the line with the doubled origin discussed in Example 5.1.4. We can see explicitly that the diagonal is not closed in this case: indeed, $X \times X$ has an atlas consisting of $U_{1,1}, U_{1,2}, U_{2,1}, U_{2,2}$ where each $U_{i,j}$ is a copy of $\mathbb{A}_k^1 \times \mathbb{A}_k^1$: the diagonal Δ in $X \times X$ restricts to the diagonal in $U_{1,1}$ and $U_{2,2}$, so is closed on these open subsets, but $\Delta|_{U_{1,2}}$ and $\Delta|_{U_{2,1}}$ are each equal to the complement of the origin in the diagonal of $\mathbb{A}_k^1 \times \mathbb{A}_k^1$, so are not closed. Thus, Δ is not closed.

Put differently, if P_1, P_2 denote the two origins in X, then we see from the above atlas on $X \times X$ that while the points (P_1, P_1) and (P_2, P_2) are in the diagonal, the points (P_1, P_2) and (P_2, P_1) are in the closure of the diagonal, but not in the diagonal.

Definition 5.3.9. A **subvariety** of a variety X is a subprevariety.

The terminology is justified by the following.

Proposition 5.3.10. *Let X be a variety, and $Y \subseteq X$ a subprevariety. Then Y is a variety.*

Proof. We have
$$\Delta(Y) = \Delta(X) \cap (Y \times Y) \subseteq X \times X.$$
Since $\Delta(X)$ is closed and $Y \times Y$ has the subset topology in $X \times X$, we conclude $\Delta(Y)$ is closed in $Y \times Y$. $\square$

Corollary 5.3.11. *Every quasiaffine variety is a variety.*

For context, we mention the following, which says in particular that we could have defined the variety condition in terms of the diagonal map giving an isomorphism onto a closed subprevariety of $Y \times Y$.

Exercise 5.3.12. Show that if Y is any prevariety, then $\Delta(Y) \subseteq Y \times Y$ is a locally closed subset, and the diagonal morphism is an isomorphism of Y onto $\Delta(Y)$.

The following will be useful for checking that a prevariety is a variety.

Proposition 5.3.13. *A prevariety X is a variety if and only if for any two points $P, Q \in X$ there is an open subset U of X which contains P and Q and is a variety.*

Proof. If X is a variety, we can take the open subset to be all of X. Conversely, suppose the condition holds; we wish to show $\Delta(X)$ is closed. Thus, suppose (P, Q) is in the closure of $\Delta(X)$. By hypothesis, we can choose U containing P and Q and such that $\Delta(U)$ is closed in $U \times U$. Since $U \times U$ has the subset topology in $X \times X$, and $(P, Q) \in U \times U$, the hypothesis that (P, Q) is in the closure of $\Delta(X)$ implies it is in the closure of $\Delta(U)$, thus in $\Delta(U) \subseteq \Delta(X)$, and since P and Q were arbitrary, we conclude $\Delta(X)$ is closed. $\square$

Warning 5.3.14. It is not necessarily the case that if X is a variety and $P, Q \in X$, then there is an affine open subset $U \subseteq X$ containing P and Q. This is closely related to the fact that not all varieties can be imbedded into projective space.

Example 5.3.15. The projective line constructed in Example 5.1.4 is a variety. In order to see this, we claim that if we let V be the open subprevariety obtained by removing the point (1) from each copy of $\mathbb{A}^1_k$ used to define $\mathbb{P}^1_k$, then $V \cong \mathbb{A}^1_k$. This will then prove that $\mathbb{P}^1_k$ is a variety using Proposition 5.3.13, since any two points are contained either in one of the two copies of $\mathbb{A}^1_k$ in the atlas for $\mathbb{P}^1_k$, or in V.

Now, V is defined by gluing two copies of $Y = \mathbb{A}^1_k \smallsetminus (0)$ to each other via the map $t \mapsto 1/t$. Denote the two copies of Y by V_1 and V_2. Recalling that Y is affine, isomorphic to $Z(x, y) \subseteq \mathbb{A}^2_k$, we see that the V_i give an atlas for V as a prevariety. To construct an isomorphism to $\mathbb{A}^1_k$, consider the regular function defined by $1/(t-1)$ on V_1, and by $1/(1/t - 1) = t/(1-t)$ on V_2. This defines a morphism to $\mathbb{A}^1_k$, with inverse morphism defined by sending x to $(x+1)/x$ in V_1 for $x \neq 0$, and to $x/(x+1)$ in V_2 for $x \neq -1$.

To illustrate the importance of the variety condition, we have the following:

Exercise 5.3.16. Let X be a prevariety and Y a variety, and $f, g : X \to Y$ two morphisms. Then the subset
$$Z = \{P \in X : f(P) = g(P)\}$$
is closed.

Exercise 5.3.17. Show that the conclusion of Exercise 5.3.16 may fail if Y is the line with the doubled origin of Example 5.1.4.

Exercise 5.3.18. Show that if Y is a variety, and $U, V \subseteq Y$ open subvarieties which are each isomorphic to affine varieties, then $U \cap V$ is also isomorphic to an affine variety. Show also that this may fail if Y is an arbitrary prevariety.

We conclude by mentioning that many of the definitions we have made previously, include local rings at a point, fields of rational functions, dimension and codimension, rational functions and maps, and nonsingularity extend immediately to prevarieties. Although the definition makes sense in general, we typically only consider rational maps and birational maps in the context of varieties, due to issues related to Exercise 5.3.17.

Many of our results so far can be reduced to the affine case, and thus extend to general prevarieties. However, some require some extra care, again as demonstrated by Exercise 5.3.17. We illustrate a few routine generalizations in the following exercises.

Exercise 5.3.19. Let X be a prevariety.

(a) If $U \subseteq X$ is open, then $\dim U = \dim X$.
(b) If $Z \subseteq X$ is closed and irreducible, then $\dim X = \dim Z + \operatorname{codim}_X Z$.

We also have the following basic fact.

Exercise 5.3.20. Let $\varphi : X \to Y$ be a morphism of varieties. Let $Z \subseteq X$ be the closure of $\varphi(X)$. Then Z is a variety, and $\dim Z \leqslant \dim X$.

CHAPTER 6

Projective varieties

We now move on to studying projective varieties, which we will treat as examples of the more general abstract varieties we have defined. We will put off the proof of one basic result: that global regular functions on a projective variety are constant. Instead of a direct algebraic proof, this result will fall out of more general geometric ideas which we will develop in connection with the notion of completeness.

6.1. Projective space

We will first construct projective space $\mathbb{P}^n_k$ as a prevariety. As a set, we define

$$\mathbb{P}^n_k = \{(a_0, \ldots, a_n) : a_i \neq 0 \text{ for at least one } i\}/\sim,$$

where $(a_0, \ldots, a_n) \sim (b_0, \ldots, b_n)$ if there exists $\lambda \in k^*$ such that $(a_0, \ldots, a_n) = \lambda(b_0, \ldots, b_n)$.

To define the topology, we recall that a polynomial $F \in k[X_0, \ldots, X_n]$ is **homogeneous** if all its monomials have the same total degree. We then observe that if $F \in k[X_0, \ldots, X_n]$ is homogeneous, although F does not define a function on $\mathbb{P}^n_k$ because of the equivalence relation, we have

$$F(\lambda a_0, \ldots, \lambda a_n) = \lambda^d F(a_0, \ldots, a_n),$$

where d is the degree of F. Thus, F has a well-defined zero set in $\mathbb{P}^n_k$, which we denote by $Z_h(F) \subseteq \mathbb{P}^n_k$. More generally, if $S \subseteq k[X_0, \ldots, X_n]$ consists entirely of homogeneous polynomials (not necessarily all of the same degree), we can define

$$Z_h(S) = \bigcap_{F \in S} Z_h(F) \subseteq \mathbb{P}^n_k.$$

Definition 6.1.1. A subset of $\mathbb{P}^n_k$ is **algebraic** if it is of the form $Z_h(S)$ for some collection S of homogeneous polynomials.

Proposition 6.1.2. *We have:*

(i) $Z_h(1) = \emptyset$ *and* $Z_h(0) = \mathbb{P}^n_k$*;*
(ii) $\bigcap_{i \in I} Z_h(S_i) = Z_h(\bigcup_{i \in I} S_i)$*;*
(iii) $Z_h(S_1) \cup Z_h(S_2) = Z_h(S_1 S_2)$.

The proof is the same as in the affine case, or can even be seen to follow from the affine case, considering zero sets inside of $\mathbb{A}^{n+1}_k$.

This means we can define the topology on $\mathbb{P}^n_k$ to have closed sets consisting precisely of algebraic subsets.

It remains to give $\mathbb{P}^n_k$ an atlas. For this, choose an index i, and observe that if we set $U_i = \mathbb{P}^n_k \smallsetminus Z_h(X_i)$, then every element of U_i has a unique representative $(a_0, \ldots, a_n)$ with $a_i = 1$. We thus define $\varphi_i : \mathbb{A}^n_k \to U_i$ by $\varphi_i(b_1, \ldots, b_n) = (b_1, \ldots, b_i, 1, b_{i+1}, \ldots, b_n)$.

To show that this gives an atlas, we need to explain a basic pair of constructions to go between homogeneous and inhomogeneous polynomials.

Notation 6.1.3. Given a polynomial $f \in k[x_1, \ldots, x_n]$ of degree d, let $h_i(f) \in k[X_0, \ldots, X_n]$ be the homogeneous degree-d polynomial obtained as follows: if $f = \sum_{(j_1,\ldots,j_n)} a_{j_1,\ldots,j_n} x_1^{j_1} \cdots x_n^{j_n}$, set

$$h_i(f) = \sum_{(j_1,\ldots,j_n)} a_{j_1,\ldots,j_n} X_0^{j_1} \cdots X_{i-1}^{j_i} X_i^{d-\sum_\ell j_\ell} X_{i+1}^{j_{i+1}} \cdots X_n^{j_n}.$$

If $F \in k[X_0, \ldots, X_n]$ is homogeneous of degree d, then let $d_i(F) \in k[x_1, \ldots, x_n]$ be obtained as follows: if $F = \sum_{(j_0,\ldots,j_n):\sum_\ell j_\ell = d} a_{j_0,\ldots,j_n} X_0^{j_0} \cdots X_n^{j_n}$, set

$$d_i(F) = \sum_{(j_0,\ldots,j_n):\sum_\ell j_\ell = d} a_{j_0,\ldots,j_n} x_1^{j_0} \cdots x_i^{j_{i-1}} x_{i+1}^{j_{i+1}} \cdots x_n^{j_n}.$$

Exercise 6.1.4. With the above notation, show:

(a) For any $f \in k[x_1, \ldots, x_n]$, we have $\varphi_i(Z(f)) = Z_h(h_i(f)) \cap U_i$.
(b) For any homogeneous $F \in k[X_0, \ldots, X_n]$, we have $\varphi_i^{-1}(Z_h(F)) = Z(d_i(F))$.

Corollary 6.1.5. *The map φ_i is a homeomorphism. Moreover, for all $i \neq j$, the map*

$$\varphi_{i,j} : \varphi_i^{-1}(U_j) \xrightarrow{\varphi_i} U_i \cap U_j \xrightarrow{\varphi_j^{-1}} \varphi_j^{-1}(U_i)$$

is a morphism.

Equivalently, $\{U_i, \varphi_i\}$ define an atlas for $\mathbb{P}_k^n$ as a prevariety.

Proof. The map φ_i is visibly bijective, so to see it is a homeomorphism, we need to check that closed subsets coincide on both sides.

Since the topologies on $\mathbb{A}_k^n$ and U_i are obtained by intersecting sets of the form $Z(f)$ and $Z_h(F) \cap U_i$ respectively, we see from Exercise 6.1.4 that φ_i is a homeomorphism.

Now, for $i \neq j$, we verify that the transition map $\varphi_{i,j} : \varphi_i^{-1}(U_j) \to \varphi_j^{-1}(U_i)$ is a morphism. But from the definitions, we see that it is given by

$$\begin{aligned}(a_1, \ldots, a_n) &\mapsto (a_1, \ldots, a_i, 1, a_{i+1}, \ldots, a_n)\\ &\mapsto \begin{cases} (a_1/a_j, \ldots, a_i/a_j, 1/a_j, a_{i+1}/a_j, \ldots, a_{j-1}/a_j, a_{j+1}/a_j, \ldots, a_n/a_j) : & i < j \\ (a_1/a_{j+1}, \ldots, a_j/a_{j+1}, a_{j+2}/a_{j+1}, \ldots, a_i/a_{j+1}, 1/a_{j+1}, a_{i+1}/a_{j+1}, \ldots, a_n/a_{j+1}) : & i > j. \end{cases}\end{aligned}$$

But we also have

$$\varphi_i^{-1}(U_j) = \begin{cases} \{(a_1, \ldots, a_n) : a_j \neq 0\} : & i < j \\ \{(a_1, \ldots, a_n) : a_{j+1} \neq 0\} : & i > j, \end{cases}$$

so we see that $\varphi_{i,j}$ is a morphism, as desired. □

Thus, we see that $\mathbb{P}_k^n$ is a prevariety. In fact, it is a variety. One can check this directly from the definitions by writing down the topology on a product of projective spaces, but we will give a different proof in Corollary 6.2.7 below.

Having defined projective space, we can now make the following definition.

Definition 6.1.6. A **projective variety** is a closed subprevariety of $\mathbb{P}_k^n$ for some n. A **quasiprojective variety** is a subprevariety of $\mathbb{P}_k^n$ for some n.

Remark 6.1.7. Since $\mathbb{A}_k^n$ is isomorphic to an open subset of $\mathbb{P}_k^n$, we see that quasiaffine varieties are all isomorphic to quasiprojective varieties.

A construction which is frequently useful in studying projective algebraic sets is the following:

Definition 6.1.8. Given $F_1, \ldots, F_m \in k[X_0, \ldots, X_n]$ homogeneous, let $Y = Z_h(F_1, \ldots, F_m)$, and assume $Y \neq \emptyset$. Then the **affine cone** over Y is $Z(F_1, \ldots, F_m) \subseteq \mathbb{A}_k^{n+1}$.

EXERCISE 6.1.9. Let Y be a nonempty algebraic set in projective space, and X the affine cone over Y.

(a) Show that X consists of the origin together with the preimage of Y under the natural map $\mathbb{A}_k^{n+1} \smallsetminus (0) \to \mathbb{P}_k^n$. In particular, there is a natural surjective map
$$\varphi : X \smallsetminus (0) \to Y.$$
(b) Show that φ is continuous.
(c) Show that φ is open (i.e., the image of an open subset of $X \smallsetminus (0)$ is an open subset of Y).
(d) Show that X is irreducible if and only if Y is irreducible.

EXERCISE 6.1.10. Let Y be a projective variety, and X the affine cone over Y.

(a) Show that $\dim \mathbb{P}_k^n = n$.
(b) Show that $\operatorname{codim}_{\mathbb{A}_k^{n+1}} X = \operatorname{codim}_{\mathbb{P}_k^n} Y$, and $\dim X = \dim Y + 1$.
(c) Conclude that if $\operatorname{codim}_{\mathbb{P}_k^n} Y = 1$, then $Y = Z_h(F)$ for some homogeneous polynomial F. Such a variety is a **projective hypersurface**.

EXERCISE 6.1.11. Let $X, Y \subseteq \mathbb{P}_k^n$ be projective varieties, and suppose that $\operatorname{codim}_{\mathbb{P}_k^n} X + \operatorname{codim}_{\mathbb{P}_k^n} Y \leqslant n$. Then show that $X \cap Y \neq \emptyset$, and every irreducible component Z of $X \cap Y$ satisfies
$$\operatorname{codim}_{\mathbb{P}_k^n} Z \leqslant \operatorname{codim}_{\mathbb{P}_k^n} X + \operatorname{codim}_{\mathbb{P}_k^n} Y.$$

6.2. Projective varieties and morphisms

We will begin by studying regular functions on quasiprojective varieties, giving a rational function description analogous to the quasiaffine case. As we have mentioned, a homogeneous polynomial F of degree d doesn't define a function on $\mathbb{P}_k^n$ (we instead refer to it as a **form**), but a quotient of two of them does – at least, where the denominator is nonvanishing. This is compatible with our affine charts as follows.

EXERCISE 6.2.1. With notation as in Exercise 6.1.4, show:

(a) For any homogeneous $F, G \in k[X_0, \ldots, X_n]$ of equal degree, the function F/G on $\mathbb{P}_k^n \smallsetminus Z(G)$ has the property that
$$(F/G) \circ \varphi_i = d_i(F)/d_i(G)$$
on $\mathbb{A}_k^n \smallsetminus Z(d_i(G))$.
(b) Given $g, h \in k[x_1, \ldots, x_n]$, the function g/h on $\mathbb{A}_k^n \smallsetminus Z(h)$ has the property that
$$(g/h) \circ \varphi_i^{-1} = X_i^d h_i(g)/h_i(h)$$
on $U_i \smallsetminus Z(h_i(h))$, where $d = \deg h - \deg g$.

COROLLARY 6.2.2. *Let $X \subseteq \mathbb{P}_k^n$ be a quasiprojective variety, $U \subseteq X$ open, and $f : U \to k$ a function. Then f is regular if and only if for all $P \in U$, there exists V an open neighborhood of P and $F, G \in k[X_0, \ldots, X_n]$ homogeneous of equal degree such that $V \cap Z_h(G) = \emptyset$ and $f = F/G$ on V.*

PROOF. It is immediate from Exercise 6.2.1 (a) that F/G defines a regular function on V. Conversely, if f is regular and $P \in U_i$, then $f \circ \varphi_i : U_i \cap U \to k$ is regular, so there is some $V_i \subseteq X_i$ a neighborhood of $\varphi_i^{-1}(P)$ on which $f \circ \varphi_i = g/h$ for some $g, h \in k[x_1, \ldots, x_n]$, with $V_i \cap Z(h) = \emptyset$. Applying Exercise 6.2.1 (b), we have $f = X_i^d h_i(g)/h_i(h)$, so we can express f in the desired form. $\square$

Next, recall that if $Y \subseteq \mathbb{A}^n_k$ is affine, then for any prevariety X, morphisms $X \to Y$ are the same as n-tuples of regular functions on X such that the induced map $X \to \mathbb{A}^n_k$ has image contained in Y. A similar, but slightly more complicated statement holds in the case that $Y \subseteq \mathbb{P}^n_k$ is projective.

In this case, if we have an $(n+1)$-tuple $(f_0, \dots, f_n)$ of regular functions on X, and if further there is no point of X at which all the f_i are zero, then we get a map $X \to \mathbb{P}^n_k$. However, not every morphism can be globally described by such an $(n+1)$-tuple. The correct statement is as follows:

Proposition 6.2.3. *Given a map $\varphi : X \to Y$, with X any prevariety and $Y \subseteq \mathbb{P}^n$ a projective variety, φ is a morphism if and only if it is described locally by $(n+1)$-tuples of regular functions which do not all vanish simultaneously.*

We first prove the following general lemma:

Lemma 6.2.4. *Let X be a prevariety, U an open subset, and $f : U \to k$ regular. Then f is a unit in $\mathscr{O}(U)$ if and only if $f(P) \neq 0$ for all $P \in U$.*

Proof. This is clear from the definitions in the case that X is affine, and the general prevariety case reduces to the affine case by the definition of regularity for prevarieties. □

Proof of Proposition 6.2.3. First suppose φ is a morphism, and let $V_j = Y \smallsetminus Z(x_j)$ for $j = 0, \dots, n$. Then on each V_j we have the regular functions induced by $\frac{x_i}{x_j}$ on $\mathbb{P}^n$ for $i = 0, \dots, n$, which gives us an $(n+1)$-tuple of regular functions on $\varphi^{-1}(V_j)$ by composing with φ. Noting that in $\mathbb{P}^n \smallsetminus Z(x_j)$ the point $(c_0, \dots, c_n)$ is also represented by $(\frac{c_0}{c_j}, \dots, \frac{c_n}{c_j})$, and that $\frac{x_i}{x_i} = 1$ vanishes nowhere on $\varphi^{-1}(V_j)$, we see that φ is represented by this $(n+1)$-tuple on $\varphi^{-1}(V_j)$. Letting j vary, we find that φ is everywhere locally represented by $(n+1)$-tuples of regular functions which do not all vanish simultaneously.

Conversely, suppose that we have some open cover U_i of X such that on each U_i, we can express φ as an $(n+1)$-tuple of regular functions which do not all vanish simultaneously. By refining the U_i, we may assume they are affine. Since being a morphism is a local condition on X, it is enough to see that if we consider each U_i as a prevariety, then φ induces a morphism $U_i \to Y$. Moreover, since $U_i \to Y$ is a morphism if and only if the composed map $U_i \to \mathbb{P}^n$ is a morphism, it suffices to treat the case $Y = \mathbb{P}^n$. Thus, we have reduced to the case that X is affine, $Y = \mathbb{P}^n$, and φ is given globally by an $(n+1)$-tuple of regular functions $f_0, \dots, f_n$ on X. But if for $j = 0, \dots, n$ we write $U_j := X \smallsetminus Z(f_j)$, then by hypothesis the U_j cover X, and each $U_j = \varphi^{-1}(\mathbb{P}^n \smallsetminus Z(x_j))$. Taking coordinates $\frac{x_i}{x_j}$ (with $i \neq j$) on $\mathbb{P}^n \smallsetminus Z(x_j)$, the map $U_j \to \mathbb{P}^n \smallsetminus Z(x_j)$ is then given by the n-tuple of functions $\frac{f_i}{f_j}$ for $i \neq j$, which are regular by Lemma 6.2.4. Then our map $U_j \to \mathbb{P}^n \smallsetminus Z(x_j)$ is a morphism for every j by Proposition 5.2.12, and because being a morphism is local on the target, we conclude that φ is a morphism. □

In the case of maps between quasiprojective varieties, we obtain the following.

Corollary 6.2.5. *Let $X \subseteq \mathbb{P}^n_k$ and $Y \subseteq \mathbb{P}^m_k$ be quasiprojective varieties. Given $F_0, \dots, F_m \in k[X_0, \dots, X_n]$ homogeneous of degree d, suppose that $Z(F_0) \cap \dots \cap Z(F_m) \cap X = \emptyset$, and the induced map $X \to \mathbb{P}^m_k$ has image contained in Y. Then the resulting map $X \to Y$ is a morphism.*

More generally, a map $X \to Y$ is a morphism if and only if everywhere locally it can be expressed as above.

Proof. First, we note that there is in fact an induced map $X \to \mathbb{P}^m_k$, since scaling the X_i by λ scales each F_i by λ^d. Now, given $P \in X$, by hypotheses there is some i such that $F_i(P) \neq 0$, so $U = X \smallsetminus Z(F_i)$ is an open neighborhood of P on which we can represent our map by $(F_0/F_i, \dots, F_m/F_i)$. We conclude from Corollary 6.2.2 that the map to Y is given locally on X by $(m+1)$-tuples of regular functions, and then Proposition 6.2.3 implies that we have a morphism, as desired.

Since the condition of being a morphism is local, one direction of the second assertion follows immediately. Conversely, if $\varphi : X \to Y$ is a morphism, Proposition 6.2.3 implies that locally φ is given by $(m+1)$-tuples of regular functions on X without simultaneous zeros. Applying Corollary 6.2.2, for every $P \in X$ there is an open neighborhood of P on which φ can be expressed as $(F_0/G_0, F_1/G_1, \dots, F_m/G_m)$ where $P \notin Z(G_i)$ for any i, for some i we have $P \notin Z(F_i)$, and $\deg F_i = \deg G_i$ for all i. Multiplying through by $G_0 \cdot G_1 \cdots G_m$ expresses φ in the desired form. $\square$

Example 6.2.6. Any linear change of coordinates is an automorphism $\mathbb{P}^n_k \xrightarrow{\sim} \mathbb{P}^n_k$. Indeed, it is given by an $(n+1)$-tuple of linearly independent homogeneous linear polynomials $(F_0, \dots, F_n)$, so the common zero set is the origin in k^{n+1}, and empty in $\mathbb{P}^n_k$. By Corollary 6.2.5, it is a morphism. But the inverse map is of the same form, so we conclude that we have an automorphism, as desired.

In particular, we see that if $H \subseteq \mathbb{P}^n_k$ is a hyperplane, then $\mathbb{P}^n_k \smallsetminus H \cong \mathbb{A}^n_k$, since there is an automorphism mapping $Z(x_0)$ to H.

We can now easily conclude our previously promised result.

Corollary 6.2.7. *Any quasiprojective variety is a variety.*

Proof. It is enough to show that $\mathbb{P}^n_k$ is a variety for any n. By our criterion for varieties (Proposition 5.3.13), it is enough to show that any two points of $\mathbb{P}^n_k$ are contained in an open subset which is a variety. But we can always find a hyperplane H which does not contain any two given points, and then $\mathbb{P}^n_k \smallsetminus H \cong \mathbb{A}^n_k$ by Example 6.2.6, so we conclude the desired statement. $\square$

Example 6.2.8. What if we take Example 6.2.6, but use fewer linear forms? Say we have $F_0, \dots, F_m$ linearly independent homogeneous linear polynomials, for some $m < n$. Then $(F_0, \dots, F_m)$ defines a rational map $\mathbb{P}^n_k \dashrightarrow \mathbb{P}^m_k$, which is a morphism away from the set $Z = Z(F_0) \cap \dots \cap Z(F_m)$. We see that Z is a linear subspace of $\mathbb{P}^n_k$, of dimension $n-m-1$. This rational map is called **linear projection** from Z. It can be described geometrically as follows: choose another linear subspace $Z' \subseteq \mathbb{P}^n_k$, of dimension m, with $Z' \cap Z = \emptyset$. Then given a suitable choice of coordinates on Z', our map can be expressed as follows: it is the morphism $\mathbb{P}^n_k \smallsetminus Z \to Z'$ which sends a point P to the point $Q \in Z'$ which is the unique intersection point of Z' with the linear span of P and Z. (Notice that the linear span of P and Z is a linear space of dimension $n-m$, so meets Z' in one point)

Example 6.2.9. Consider $Y = Z_h(X_0X_1 - X_2^2) \subseteq \mathbb{P}^2_k$. We claim that $Y \cong \mathbb{P}^1_k$. We can construct a morphism $Y \to \mathbb{P}^1_k$ by linear projections as follows: away from $(0,1,0)$, we have $(X_0, X_1, X_2) \mapsto (X_0, X_2)$, and away from $(1,0,0)$, we have $(X_0, X_1, X_2) \mapsto (X_2, X_1)$. On Y, away from both $(0,1,0)$ and $(1,0,0)$ we have $X_0/X_2 = X_2/X_1$, so we see that these two maps together yield a morphism $Y \to \mathbb{P}^1_k$. The inverse is given by $(Y_0, Y_1) \mapsto (Y_0^2, Y_1^2, Y_0Y_1)$, so we get the claimed isomorphism.

Remark 6.2.10. Most varieties which are typically studied are quasiprojective, either by construction, or by coincidence. We will show in §7.2 that every nonsingular curve is (isomorphic to) a quasiprojective variety. This turns out to be true also for singular curves, and for nonsingular surfaces. However, it is possible to construct singular abstract surfaces and nonsingular abstract three-dimensional varieties which are not isomorphic to any quasiprojective variety. For one such construction, see Example 3.4.1 of Appendix B of [**Har77**]; one can also construct examples using toric varieties.

As in the proof of Corollary 6.2.7, a quasiprojective variety has the property that any finite collection points may be placed in a common affine open subset, so one way to show a variety is not quasiprojective is to produce a finite set of points such that any affine open subset cannot contain all of them. Conversely, a theorem of Kleiman asserts that if a nonsingular complete (see §8.2) variety has the property that any finite set of points can be placed in a common affine open subset, then the variety is projective.

Exercise 6.2.11. Let Y be a projective algebraic set, and X the affine cone over it. Show that the map $\varphi : X \smallsetminus (0) \to Y$ studied in Exercise 6.1.9 is a morphism.

6.A. Homogeneous ideals and coordinate rings

Projective varieties have a theory of ideals and coordinate rings parallel to the affine case, although there are some crucial differences (see particularly Exercise 6.A.8 below). We will not make use of this material elsewhere, but it is fundamental to understanding projective varieties, so we include a summary. In this appendix, let $B_n = k[X_0, \ldots, N_n]$.

Definition 6.A.1. An ideal of B_n is **homogeneous** if it can be generated by homogeneous elements.

Exercise 6.A.2. An ideal $I \subseteq B_n$ is homogeneous if and only if it has the following property: for every $f \in I$, if we write $f = F_0 + F_1 + \cdots + F_d$ as a sum of homogeneous polynomials with F_i of degree i, then each $F_i \in I$.

Definition 6.A.3. Given $S \subseteq \mathbb{P}^n_k$, let $I(S)$ be the (necessarily homogeneous) ideal generated by $F \in B_n$ homogeneous such that $F(P) = 0$ for all $P \in S$.

Note that we have to take the ideal generated by the F as in the definition because the condition only makes sense when F is homogeneous, while no (nonzero) ideal consists entirely of homogeneous elements.

Exercise 6.A.4. Given $S \subseteq \mathbb{P}^n_k$, the ideal $I(S)$ is radical.

Definition 6.A.5. If $I \subseteq B_n$ is a homogeneous ideal, let $Z_h(I) \subseteq \mathbb{P}^n_k$ be the set of $P \in \mathbb{P}^n_k$ such that $F(P) = 0$ for all homogeneous $F \in I$.

The fundamental ideal-variety correspondence is then:

Exercise 6.A.6. There is a one-to-one, inclusion-reversing correspondence

$$\begin{array}{rcl} \{\text{algebraic sets } Y \subseteq \mathbb{P}^n_k\} & \longleftrightarrow & \{\text{homogeneous radical ideals } I \subseteq B_n, I \neq (X_0, \ldots, X_n)\} \\ & \text{given by} & \\ Y & \longmapsto & I(Y) \\ Z_h(I) & \longleftarrow\!\shortmid & I. \end{array}$$

Moreover, under this correspondence, varieties correspond to prime ideals.

In the above correspondence, we do not allow $I = (X_0, \ldots, X_n)$, since it corresponds to the empty set, just as $I = B_n$ does. We choose this convention so that the prime ideals correspond precisely to projective subvarieties, which are by definition nonempty.

This motivates the following definition:

Definition 6.A.7. If $Y \subseteq \mathbb{P}^n_k$ is an algebraic set, the **homogeneous coordinate ring** $H(Y)$ of Y is given by $H(Y) = B_n/I(Y)$.

Thus, if X is the affine cone over Y, the homogeneous coordinate ring of Y is simply equal to $A(X)$. However, homogeneous coordinate rings behave quite differently from affine coordinate rings. Morphisms of projective varieties need not induce homomorphisms of their homogeneous coordinate rings, and in particular homogeneous coordinate rings are not invariant under isomorphisms of projective varieties.

Exercise 6.A.8. Show that the isomorphic projective varieties considered in Example 6.2.9 (the projective line and conic plane curve) do not have isomorphic homogeneous coordinate rings.

The above should not be too surprising: after all, different imbeddings of a given projective variety could have very different geometry near the origin of their affine cones.

The key property of homogeneous coordinate rings is that, because they are obtained by modding out by a homogeneous ideal, they inherit the same notion of degree enjoyed by the polynomial ring B_n. That is, there is a notion of elements of $H(Y)$ being homogeneous of degree d, and every element can be written uniquely as a sum of homogeneous elements of different degrees. We then have the following descriptions of local rings and the function field of a projective variety.

Exercise 6.A.9. Let X be a projective variety. Then:

(a) For $P \in X$, let $\mathfrak{p}$ be the corresponding prime ideal of $H(X)$. Then the local ring $\mathscr{O}_{P,X}$ can be identified with the subring of the local ring $H(X)_{\mathfrak{p}}$ consisting of elements expressible as $\frac{f}{g}$ for $f, g \in H(X)$ homogeneous of equal degree.

(b) The function field $K(X)$ can be identified with the subfield of the fraction field of $H(X)$ consisting of elements expressible as $\frac{f}{g}$ for $f, g \in H(X)$ homogeneous of equal degree.

CHAPTER 7

Nonsingular curves

The primary goal of this chapter is to prove that every abstract nonsingular curve can be realized as an open subset of a (unique) nonsingular projective curve. Note that this encapsulates two facts in one: that every nonsingular abstract curve is quasiprojective, and that it can be "compactified" into a projective curve without introducing singularities.

We start from the definitions and state the necessary background algebra.

7.1. Curves, regular functions, and morphisms

Our discussion of the abstract definition of a variety allows us to work transparently with abstract curves.

Definition 7.1.1. A **curve** is a variety of dimension 1.

We first show the following.

Lemma 7.1.2. *Let P be a nonsingular point of a curve C. Then there exists an open neighborhood U of P and a regular function t on U such that $t(Q) \neq 0$ for all $Q \in U \smallsetminus \{P\}$, and for all $V \subseteq U$ open containing P, and $f \in \mathscr{O}(V)$, we have $f(P) = 0$ if and only if t divides f in $\mathscr{O}(V)$.*

Consequently, for all $f \in \mathscr{O}(V \smallsetminus \{P\})$, we can write $f = t^\nu g$, where $\nu \in \mathbb{Z}$, and $g \in \mathscr{O}(V)$ satisfies $g(P) \neq 0$. Moreover, ν is uniquely determined and independent of choice of t, and $f \in \mathscr{O}(V) \subseteq K(C)$ if and only if $\nu \geqslant 0$.

Proof. By definition of nonsingular, the maximal ideal $\mathfrak{m}_P \subseteq \mathscr{O}_{P,C}$ can be generated by $\dim C = 1$ elements, so let t be any generator. Then t is regular on some open neighborhood of P; if t is regular on some U', then t has finitely many zeroes on U'. Let U be the complement of the zeroes of t other than P. Then t vanishes only at P when considered as a regular function on U. We claim that this implies the desired statement: given $V \subseteq U$ and $f \in \mathscr{O}(V)$, certainly if t divides f in $\mathscr{O}(V)$, then $f(P) = 0$, but conversely, if $f(P) = 0$, then $f \in \mathfrak{m}_P$, so t divides f in $\mathscr{O}_{P,C}$. This implies that f/t determines a regular function on some neighborhood of P. But since $t(Q) \neq 0$ for all $Q \in U \smallsetminus \{P\}$, we have that t is a unit in $\mathscr{O}(U \smallsetminus \{P\})$, so f/t is regular on $V \smallsetminus \{P\}$. Thus, f/t is regular on an open cover of V, and hence it is regular on V, and we conclude that t divides f in $\mathscr{O}(V)$, as desired.

For the second assertion, if f is regular on V, then we can inductively divide out by t in $\mathscr{O}_{P,C}$ until we obtain $g(P) \neq 0$; this process must eventually terminate because $\mathscr{O}_{P,C}$ is Noetherian. In this case, we see that $\nu \geqslant 0$. If f is not regular on V, it is still an element of $K(C)$, which is the fraction field of $\mathscr{O}_{P,C}$. Thus, if $f = h_1/h_2$ with $h_1, h_2 \in \mathscr{O}_{P,C}$, then using the regular case on h_1 and h_2 we get that in $\mathscr{O}_{P,C}$, we can write $f = t^\nu h_3/h_4$ for some $\nu \in \mathbb{Z}$, and $h_3, h_4 \in \mathscr{O}_{P,C} \smallsetminus \mathfrak{m}_P = \mathscr{O}_{P,C}^*$. Then we can set $g = h_3/h_4$; this is in $\mathscr{O}_{P,C}$, so is regular in a neighborhood of P, but we also have $g = ft^{-\nu}$, so is regular on $V \smallsetminus \{P\}$ as well. This gives the desired assertion.

Now, if we have $f = t^{\nu_1} g_1 = t^{\nu_2} g_2$, with $\nu_1 \geqslant \nu_2$, then $g_2 = t^{\nu_1 - \nu_2} g_1$, and since $g_2 \in \mathscr{O}_{P,C}^*$, but $t \in \mathfrak{m}_P$ and $g_1 \in \mathscr{O}_{P,C}$, we must have $\nu_1 - \nu_2 = 0$, as desired. Now, if $\nu \geqslant 0$ then obviously $f \in \mathscr{O}(V)$. Conversely, if $\nu < 0$, then $ft^{-\nu} = g_1$, and since $g_1 \notin \mathfrak{m}_P$, we conclude that we cannot have $f \in \mathscr{O}_{P,C}$. Finally, if we have two choices t_1, t_2, we claim that $t_1 = t_2 g$ for some $g \in \mathscr{O}_{P,C}^*$;

it then follows immediately that ν is independent of the choice of t. We have $t_1 = t_2^{\nu_1} g_1$ for some $g_1 \in \mathscr{O}_{P,C}^*$, and also $t_2 = t_1^{\nu_2} g_2$ for some $g_2 \in \mathscr{O}_{P,C}^*$, with both ν_1, ν_2 positive since t_1 and t_2 are regular and vanishing at P. Then $t_1 = t_1^{\nu_1\nu_2} g_2^{\nu_2} g_1$, so $t_1^{\nu_1\nu_2 - 1} \in \mathscr{O}_{P,C}^*$, and we must have $\nu_1\nu_2 = 1$, and hence $\nu_1 = \nu_2 = 1$, as desired. □

REMARK 7.1.3. The argument can be expressed also in terms of discrete valuation rings – if P is a nonsingular point of a curve C, then $\mathscr{O}_{P,C}$ is a DVR.

In the second part of the lemma, we intuitively think of ν as being the order of the zero (if positive) or pole (if negative) of f at P, so we make the following definition:

DEFINITION 7.1.4. The ν of Lemma 7.1.2 is the **order** of f at P, denoted $\mathrm{ord}_P(f)$.

A fundamental result on morphisms from curves to projective varieties is the following.

THEOREM 7.1.5. *If C is a curve, and $P \in C$ a nonsingular point, and Y a projective variety, then every morphism $C \smallsetminus \{P\} \to Y$ extends uniquely to a morphism $C \to Y$.*

REMARK 7.1.6. Exercise 5.3.16 implies that the uniqueness in the theorem is satisfied for Y any variety, as a consequence of the condition analogous to being Hausdorff which we used to distinguish varieties among prevarieties. Thus, we need only to prove the existence statement for the theorem.

PROOF OF THEOREM 7.1.5. By Remark 7.1.6, it suffices to prove the existence of the desired extension. Let $\varphi : C \smallsetminus \{P\} \to Y$ be the given morphism. Let $U \ni P$ be an open subset such that there exists a t as in Lemma 7.1.2, and such that on $U \smallsetminus \{P\}$, we can represent φ by an $(n+1)$-tuple of regular functions $f_0, \dots, f_n \in \mathscr{O}(U \smallsetminus \{P\})$ which do not simultaneously vanish anywhere on $\mathscr{O}(U \smallsetminus \{P\})$.

By Lemma 7.1.2, we can write each f_i as $t^{e_i} g_i$, where $e_i = \mathrm{ord}_P(f_i) \in \mathbb{Z}$ and g_i is regular on U, with $g_i(P) \neq 0$. Choose j with e_j minimal; then since $(f_0, \dots, f_n)$ represents φ on $U \smallsetminus \{P\}$, and t is nonvanishing on this subset, scaling simultaneously by t^{-e_j} we find that $(t^{e_0 - e_j} g_0, \dots, t^{e_n - e_j} g_n)$ also represents φ on the same subset. But $e_i \geqslant e_j$ for all i, so these functions are regular on all of U, and $t^{e_j - e_j} g_j$ is non-zero at P, so setting $\varphi(P) = (t^{e_0 - e_j} g_0(P), \dots, t^{e_n - e_j} g_n(P))$ gives an extension of φ to U.

Finally, being a morphism is a local condition, so if we have extended φ to a morphism on U, since it was already a morphism on $C \smallsetminus \{P\}$, we conclude that we have extended φ to a morphism on all of C. □

REMARK 7.1.7. Theorem 7.1.5 generalizes also to the case of higher-dimensional varieties as follows: if X is a variety with the property that the singular points of X have every component of codimension at least 2 in X, and Y is a projective variety, then a morphism $\varphi : U \to Y$ for $U \subseteq X$ open can always be extended to an open subset $V \subseteq X$ such that every component of $X \smallsetminus V$ has codimension at least 2. The argument is similar, but a bit more involved. .

COROLLARY 7.1.8. *A birational map between two nonsingular projective curves extends uniquely to an isomorphism.*

PROOF. We have an isomorphism of open subsets, but by the theorem each map extends to a morphism on the whole curve. By Exercise 5.3.16, these extended morphisms must still be inverse to one another, since their compositions are the identity on open subsets. □

REMARK 7.1.9. The idea of extending morphisms of nonsingular curves as in the theorem plays an important role in algebraic geometry, more or less replacing the use of limits in metric topology. The usual notion of limit doesn't make sense for us. However, if we have $c \in (a, b) \subseteq \mathbb{R}$, and $\varphi : (a, b) \smallsetminus c \to Y$ for some space Y, then the statement that $\lim_{x \to c} \varphi(x) = Q$ for some $Q \in Y$ is equivalent to saying that φ can be extended continuously at c by setting $\varphi(c) = Q$. If we assume that φ was continuous to start with, this is the same as saying that φ remains continuous if we set $\varphi(c) = Q$.

This is the idea which we translate into algebraic geometry: we replace (a, b) and the point c by a nonsingular curve and the point P, and we replace continuous maps by morphisms. From this point of view, Theorem 7.1.5 says that in projective varieties, limits always exist. Intuitively, this is saying that projective varieties are compact; we will explore this idea in more detail when we discuss complete varieties.

Exercise 7.1.10. Assume that k doesn't have characteristic 2, and consider the projective curve $X = Z(X_1^2 X_2 - X_0^3 + X_0 X_2^2)$.

(a) Show that X is nonsingular.
(b) Show that X is not birational to $\mathbb{P}^1_k$.

7.2. Quasiprojectivity

We will now prove the following theorem:

Theorem 7.2.1. *If C is a nonsingular curve, then C is quasiprojective.*

For the proof, we need one key background statement, which we organize into an exercise.

Exercise 7.2.2. (a) Show that if $\varphi : X \to Y$ is a morphism of varieties, and $U \subseteq X$ is an open subset such that the composition $U \to Y$ is an isomorphism, then $U = X$.

(b) Show that if $\varphi : X \to Y$ is a morphism of varieties, and $U \subseteq X$ is an open subset such that $\varphi : U \to Y$ is an isomorphism onto an open subset $V \subseteq Y$, then $\varphi^{-1}(V) = U$.

(c) Give an example to demonstrate that this is false if X is allowed to be an arbitrary prevariety.

Proof of Theorem 7.2.1. Let U_i be a cover of C by affine open subsets. Then we have $U_i \subseteq \mathbb{A}^{n_i} \subseteq \mathbb{P}^{n_i}$, so we take Y_i to be the closure of U_i in $\mathbb{P}^{n_i}$. Thus Y_i is projective, and U_i is isomorphic to an open subset of Y_i. By Theorem 7.1.5, we obtain unique extensions $\varphi_i : C \to Y_i$ for each i (note that each U_i may omit more than one point of C, but we can apply the theorem inductively to extend over each one). These extensions may not be isomorphisms onto their images, because we have little control over what happens when we take the closure of U_i. The trick is to take the product over all i; we then have an induced morphism $\varphi : C \to \prod_i Y_i \subseteq \prod_i \mathbb{P}^{n_i}$. Let $Y \subseteq \prod_i Y_i$ be the closure of the image of C. We will show that C is isomorphic to an open subset of Y. This will prove the theorem, because Y is a closed subset of $\prod_i \mathbb{P}^{n_i}$, which is itself projective via the Segre imbedding (see Exercises 2.14, 3.16 of [**Har77**]).

Our first task is to show that φ is a homeomorphism onto an open subset of Y. Now, φ is injective, since given $P, Q \in C$, if $P \in U_i$, we claim $\varphi_i(P) \neq \varphi_i(Q)$. If $Q \in U_i$ as well, this follows from the injectivity of φ_i on U_i, but if $Q \notin U_i$, then $\varphi_i(Q) \notin U_i \subseteq Y_i$ by Exercise 7.2.2 (b), while $\varphi_i(P) \in U_i$, proving the claim, and thus injectivity. It then follows that φ is a homeomorphism onto its image, since φ maps finite sets to finite sets and thus closed subsets of C to closed subsets of $\varphi(C)$. We next observe that φ is dominant onto Y by definition, so Y is irreducible, and we have $K(Y_i) \hookrightarrow K(Y) \hookrightarrow K(C)$. But $C \to Y_i$ is birational, to $K(Y_i) = K(C)$, and we conclude that $K(Y) = K(C)$, so $\varphi : C \to Y$ is birational. In particular, we conclude that $K(Y)$ has transcendence degree 1, so Y is a curve, and also that $\varphi(C)$ contains an open subset of Y. But since Y is a curve, any subset containing a nonempty open subset is open, so $\varphi(C)$ is open, and we have proved that φ induces a homeomorphism from C onto an open subset of Y.

We now want to see that φ is an isomorphism of C onto its image. It suffices to show that the induced maps on local rings are isomorphisms at every point of C, so let $P \in C$, and consider the induced map $\mathscr{O}_{\varphi(P),Y} \to \mathscr{O}_{P,C}$. This is injective since $C \to Y$ is dominant. Choose i with

$P \in U_i$. Then the map $\varphi_i : U_i \to Y_i$ is an isomorphism onto its image, so the induced map $\mathscr{O}_{\varphi_i(P),Y_i} \to \mathscr{O}_{P,U_i}$ is an isomorphism. But we can factor φ_i as $U_i \hookrightarrow C \to Y \to Y_i$, where the last morphism is projection onto the ith factor from the product. This means that the map $\mathscr{O}_{\varphi_i(P),Y_i} \to \mathscr{O}_{P,U_i}$ factors through $\mathscr{O}_{\varphi(P),Y} \to \mathscr{O}_{P,C} = \mathscr{O}_{P,U_i}$, so we conclude that the latter must be surjective, and hence an isomorphism, as desired. □

REMARK 7.2.3. In fact, as mentioned in Remark 6.2.10, the theorem holds without the nonsingularity hypothesis, but the proof is a bit more involved. One approach is to show that even on a singular curve, one has an affine open cover such that for every open subset, the omitted points are nonsingular. Given that, the above argument goes through unmodified.

Now that we know that every nonsingular curve is quasiprojective, we can consider the question of projectivity. Obviously, not every nonsingular curve is projective. But we now see that every nonsingular curve can be "compactified" as an open subvariety of a projective curve by imbedding in projective space and taking the closure. But this closure will not in general be nonsingular. So we can ask whether every nonsingular curve can be realized as an open subvariety of a *nonsingular* projective curve. For the moment, although it is clear that every curve is birational to a nonsingular curve, and also to a projective curve, it is not even clear that every curve is birational to a nonsingular projective curve. We will prove the stronger assertion, but only after a discussion of normalization.

7.3. Normality and normalization

We make a brief detour to discuss the notion of normality. Most of the proofs, while not necessarily difficult, are purely algebraic, and we omit them.

DEFINITION 7.3.1. A variety X is **normal** if it is covered by affine open subvarieties U_i such that each $A(U_i)$ is integrally closed in $K(X)$.

Recall that given an inclusion of integral domains $R \subseteq S$, an element $s \in S$ is **integral** over R if it is a root of a monic polynomial with coefficients in R. We say R is **integrally closed** in S if every element of S which is integral over R is in fact an element of R. The integers are integrally closed in the rational numbers, by Gauss's lemma, motivating the terminology.

Normality is a somewhat subtle condition, but it does have a fairly direct relationship to nonsingularity. Specifically, we have:

THEOREM 7.3.2. *A nonsingular variety is normal. The singular locus of a normal variety has codimension at least* 2.

The proof of the first statement is difficult,[1] while the second is more straightforward; see §II.5 Theorems 1 and 3 of [**Sha94a**]. Since any non-empty closed subset of a curve has codimension at most 1, we conclude:

COROLLARY 7.3.3. *A normal curve is nonsingular.*

Note, however, that the corollary is quite a bit easier than Theorem 7.3.2, amounting to the basic algebra of discrete valuation rings.

Another basic algebra statement is:

PROPOSITION 7.3.4. *X is normal if and only if every affine open subset U has $A(U)$ integrally closed in $K(X)$, if and only if $\mathscr{O}_{P,X}$ is integrally closed in $K(X)$ for all points $P \in X$.*

[1] More precisely, the proof that the local ring at a nonsingular point is a unique factorization domain is difficult, while the deduction that a unique factorization domain is integrally closed is quite short – see Proposition 4.10 of [**Eis95**].

Remark 7.3.5. Although it is not our intent to give a complete account of normality, we mention some basic facts for the sake of context. First, the converse to Theorem 7.3.2 holds for hypersurfaces: if their singular locus has codimension at least 2, then they are normal – see Proposition 2 of §III.8 of [**Mum99**].

Thus, a variety such as the cone $Z(xy - z^2) \subseteq \mathbb{A}^3_k$ is normal, being a surface with a unique singularity at $(0, 0, 0)$. On the other hand, the surface $Z(w^2y - x^2, w^3z - x^3, y^3 - z^2) \subseteq \mathbb{A}^4_k$, which likewise has $(0, 0, 0, 0)$ as its only singularity, is not normal – see Example K (B) of §III.8 of [**Mum99**].

Finally, although this doesn't characterize normality, an important property of normality is that a normal variety doesn't have multiple "branches" meeting at a point (i.e., it does not look like a node, or any higher-dimensional analogue).

We will next consider *normalization* – the process of replacing a non-normal variety with a normal one. It turns out that this can be done in a canonical way. The affine version of this process is as follows: if an integral domain R is not integrally closed in its field of fractions, we can take the integral closure, which is the set of all elements of the field which are integral over R. More generally, if $R \subseteq S$ is not integrally closed in S, we can take its integral closure in S. It is a basic algebra fact (Theorem 4.2 of [**Eis95**]) that the integral closure is again a subring, and it is integrally closed in the field of fractions. One proof is related to the fact that f is integral over R if and only if the ring $R[f]$ is a finitely-generated R-module. More difficult is the theorem of Emmy Noether (Theorem 4.14 of [**Eis95**]) that if R is a finitely-generated k-algebra which is an integral domain, and L a finite extension of the fraction field of R, then the integral closure of R in L is still finitely generated over k. Putting these statements together gives normalization in the affine case.

In order to apply the affine case, we will want to use the following algebra results as well:

Proposition 7.3.6. *Let $R \subseteq S$ be an integral ring extension. Then every maximal ideal $\mathfrak{m}$ of R is realized as the intersection of R with a maximal ideal of S. Moreover, if $\mathfrak{m} = \mathfrak{p} \cap R$ for some prime ideal $\mathfrak{p} \subseteq S$, then $\mathfrak{p}$ is maximal.*

Recall that $R \subseteq S$ is **integral** if every element of S is a root of a monic polynomial with coefficients in R. This result is more or less the "going up" theorem in commutative algebra; see Proposition 4.15 and Corollary 4.17 of [**Eis95**].

Proposition 7.3.7. *Given a ring extension $R \subseteq S$ and $f \in R$ nonzero, if R' is the integral closure of R in S, then $(R')_f$ is the integral closure of R_f in S_f.*

See Proposition 4.13 of [**Eis95**].

The following fact is frequently useful.

Exercise 7.3.8. Let X be a prevariety, and U, V affine open subsets. Then $U \cap V$ can be covered by open subsets which are each simultaneously of the form $U \smallsetminus Z(f)$ for $f \in A(U)$ and $V \smallsetminus Z(g)$ for $g \in A(V)$.

We will also want to know the following.

Exercise 7.3.9. If R is an integral domain with fraction field K, and L an algebraic extension of K, then the integral closure of R in L has fraction field L.

Exercise 7.3.10. Let $\varphi : X \to Y$ be a morphism, with Y a variety and X a prevariety.

(a) Show that if $P, Q \in X$ have (P, Q) in the closure of $\Delta(X)$, then $\varphi(P) = \varphi(Q)$.
(b) Suppose that there is an open cover $\{V_i\}$ of Y such that each V_i is isomorphic to an affine variety, and also each $\varphi^{-1}(V_i)$ is isomorphic to an affine variety. Show that X is a variety.

We are now ready to define the normalization. It will be useful to give a slightly more general form than is immediately necessary.

Definition 7.3.11. If X is a prevariety, and L is a finite field extension of $K(X)$ (in particular, algebraic over $K(X)$) the **normalization** $\nu_L : \widetilde{X}_L \to X$ of X in L is the prevariety constructed as follows: if $\{\varphi_i : X_i \to U_i\}$ is an atlas for X, for each i let $\widetilde{X}_{L,i}$ be the affine variety with coordinate ring equal to the integral closure of $A(X_i)$ in L, and ν_i the corresponding morphism to X_i. For i, j, set

$$U_{i,j} = \nu_i^{-1}(\varphi_i^{-1}(U_j)),$$

and let $\varphi_{i,j} : U_{i,j} \to U_{j,i}$ be the isomorphism induced by the identifications $K(\widetilde{X}_{L,i}) = L$ (Exercise 7.3.9). Then let $\widetilde{X}_L$ be the prevariety obtained by gluing the $\widetilde{X}_{L,i}$ (Exercise 5.1.5). Let ν_L be the morphism induced by the ν_i.

In particular, the **normalization** $\nu : \widetilde{X} \to X$ of X is the normalization of X in $K(X)$.

Proposition 7.3.12. *Definition 7.3.11 describes a well-defined prevariety $\widetilde{X}_L$ and morphism $\nu_L : \widetilde{X}_L \to X$. Moreover, ν_L is surjective, with finite fibers, and the induced map on function fields is $K(X) \hookrightarrow K(\widetilde{X}_L) = L$.*

If $U_i \subseteq X$ is part of the atlas for X, then $\nu_L^{-1}(U_i)$ is the $\widetilde{U}_i$ used to define $\widetilde{X}_L$.

Finally, if X is a variety, then $\widetilde{X}_L$ is a variety.

The proof is fairly straightforward using Proposition 7.3.7 to show that the $\varphi_{i,j}$ are indeed (iso)morphisms, Proposition 7.3.6 to check surjectivity and finiteness of fibers, and Exercise 7.3.10 to see that if X is a variety, then $\widetilde{X}_L$ is also a variety.

Note that in particular, the normalization of X yields a birational morphism.

Remark 7.3.13. We start to see the utility of having a notion of abstract variety: while it is true that the normalization of a projective variety is projective, the proof isn't trivial, and something of a distraction from the basic idea, that we are simply gluing together integral closures. *A priori*, the construction of the normalization depends on a choice of atlas, but we will see in 7.3.15 below that in fact it is independent.

We see that normalization is universal for (dominant) morphisms from normal varieties:

Proposition 7.3.14. *Suppose $\varphi : Y \to X$ is a dominant morphism, with Y normal. Then φ factors through the normalization map $\widetilde{X} \to X$.*

More generally, if $L/K(X)$ is a finite extension, and we are given an inclusion $L \hookrightarrow K(Y)$ of extensions of $K(X)$, then φ factors through the normalization $\widetilde{X}_L \to X$.

Proof. Let $\{\varphi_i : X_i \to U_i\}$ be the atlas of X used to define the normalization; we will identify X_i with U_i. Also let V_j be an affine open cover of Y such that each V_j is contained in some $\varphi^{-1}(U_i)$. Let $\widetilde{U}_i$ be the preimages of U_i in $\widetilde{X}_L$. Fix i, j with $V_j \subseteq \varphi^{-1}(U_i)$. Note that the dominance of φ implies that we have an induced inclusion $A(U_i) \hookrightarrow A(V_j)$. Also $A(U_i) \subseteq A(\widetilde{U}_i)$ by definition. Using the inclusion $L \hookrightarrow K(Y)$, we can consider both these inclusions to hold inside $K(Y)$.

We claim that in fact $A(\widetilde{U}_i) \subseteq A(V_j)$ in $K(Y)$. This follows immediately from the definitions; $A(\widetilde{U}_i)$ is the set of elements of L which are integral over $A(U_i)$, and by the hypothesis that V_i is normal implies that $A(V_j)$ contains all elements of $K(Y)$ which are integral over $A(V_j)$. But because $L \subseteq K(Y)$ and $A(U_i) \subseteq A(V_j)$, any element of L integral over $A(U_i)$ is in particular an element of $K(Y)$ integral over $A(V_j)$, so must lie in $A(V_j)$. We conclude that $A(\widetilde{U}_i) \subseteq A(V_j)$, so the morphisms $V_j \to U_i$ factor through $\widetilde{U}_i \to U_i$. Thus, for each V_j we get that $V_j \to X$ factors through $\widetilde{X}_L$.

But given j, j' the resulting morphisms $V_j \to \widetilde{X}_L$ and $V_{j'} \to \widetilde{X}_L$ both induce the same inclusion $L \to K(Y)$ of function fields by construction, so they define the same rational maps, and agree on $V_j \cap V_{j'}$. We can thus glue them all together to obtain the desired morphism $Y \to \widetilde{X}_L$. $\square$

Corollary 7.3.15. *The normalization $\widetilde{X}_L$ of X in L is independent of the choice of atlas.*

If $U \subseteq X$ is open, and $\widetilde{X}_L \to X$ and $\widetilde{U}_L \to U$ the respective normalizations, then $\widetilde{U}_L$ is naturally identified with the preimage of U in $\widetilde{X}_L$.

Proof. Let $\nu_L : \widetilde{X}_L \to X$ and $\nu'_L : \widetilde{X}'_L \to X$ be normalizations of X with respect to two different atlases. Then Proposition 7.3.14 implies that ν_L and ν'_L factor through one another, necessarily inducing the identity on L. It follows that $\widetilde{X}_L$ and $\widetilde{X}'_L$ are isomorphic, via an isomorphism commuting with ν_L and ν'_L.

The second statement then follows by construction and Proposition 7.3.12, since given an atlas on X and $U \subseteq X$ open and affine, we can always make a new atlas by adding a chart which uses U. $\square$

Example 7.3.16. Consider the cuspidal curve $C \subseteq \mathbb{A}^2$ given by $y^2 = x^3$. This is a singular curve, so not normal. We have studied the morphism $\mathbb{A}^1 \to C$ given by $t \mapsto (t^2, t^3)$, corresponding to the injective homomorphism $k[x, y]/(y^2 - x^3) \to k[t]$ sending x to t^2 and y to t^3. This homomorphism induces an isomorphism on fraction fields (this follows from the observation t is the image of $\frac{y}{x}$). We see that t is integral over $A(C)$, since it satisfies $z^2 - x$ (and also $z^3 - y$). But $k[t]$ is integrally closed in its fraction field (one may check this directly, or invoke that nonsingularity implies normality), so we conclude that the morphism $\mathbb{A}^1 \to C$ is in fact the normalization of C.

Remark 7.3.17. In our discussion of normality and normalization, it is important that we are working with (pre)varieties, and not arbitrary algebraic sets. Otherwise, we wouldn't have the field of functions in which to consider integral closure.

On the other hand, it turns out to be useful to work with normalizations also for reducible algebraic sets, and in this case one uses the convention that the normalization should be the disjoint union of the normalizations of the irreducible components.

7.4. Projective curves

The main utility of normalization for us is that it provides a method of desingularizing curves, and we see that it preserves projectivity.

Theorem 7.4.1. *The normalization of a projective curve is a nonsingular projective curve.*

Proof. Let C be the projective curve, and $\widetilde{C}$ its normalization. Nonsingularity of $\widetilde{C}$ is immediate from Corollary 7.3.3, while we know that $\widetilde{C}$ is quasiprojective from Theorem 7.2.1. We claim that if we have $\widetilde{C} \subseteq \mathbb{P}^n$, it must be closed, so that $\widetilde{C}$ is projective, as desired. Let Y be the closure of $\widetilde{C}$ in $\mathbb{P}^n$. Given $P \in Y$, let U be an affine neighborhood of P in Y, and $\widetilde{U}$ its normalization. Then since $\widetilde{U}$ is a nonsingular curve and C is projective, by Theorem 7.1.5 the birational map $\widetilde{U} \dashrightarrow C$ induced by

$$\widetilde{U} \to U \hookrightarrow Y \dashrightarrow \widetilde{C} \to C$$

extends to a morphism $\widetilde{U} \to C$.

Because $\widetilde{U}$ is normal, it follows from Proposition 7.3.14 that this morphism factors through $\widetilde{C} \to C$. By construction, the induced morphism $\widetilde{U} \to \widetilde{C} \hookrightarrow Y$ agrees with the composed morphism $\widetilde{U} \to U \hookrightarrow Y$ on an open subset, and hence on all of $\widetilde{U}$. But by surjectivity of normalization, there is some $\widetilde{P} \in \widetilde{U}$ mapping to $P \in U$, and if we let Q be its image in $\widetilde{C}$, we conclude $Q = P$ in Y, so $P \in \widetilde{C}$. Since P was arbitrary in Y, we conclude that $\widetilde{C} = Y$, and $\widetilde{C}$ is projective, as desired. $\square$

REMARK 7.4.2. It is true more generally that the normalization of a projective variety is again projective, but this is a harder result.

COROLLARY 7.4.3. *Let C be a nonsingular curve. Then there is a nonsingular projective curve $\bar{C}$ (necessarily unique) such that C is isomorphic to an open subvariety of $\bar{C}$.*

PROOF. By Theorem 7.2.1, we can realize C as a quasiprojective curve. Let Y be its closure in projective space. By Theorem 7.4.1, if $\bar{C}$ is the normalization of Y, it is a projective nonsingular curve. Finally, since C is a nonsingular open subset of Y (and using the assertion on restriction to open subsets in Theorem 7.3.15), the normalization map $\bar{C} \to Y$ is an isomorphism on C, so we have that C is isomorphic to an open subvariety of $\bar{C}$, as desired. □

COROLLARY 7.4.4. *Every curve is birational to a unique nonsingular projective curve.*

PROOF. The uniqueness is Corollary 7.1.8. Given any curve C, we know it has a non-empty open subset U of nonsingular points, so applying Theorem 7.4.3 to U, we can imbed it into a nonsingular projective curve, which is then birational to C. □

We can rephrase what we've done in more abstract language as follows:

COROLLARY 7.4.5. *The following categories are equivalent:*

(a) *Projective nonsingular curves, and nonconstant morphisms between them;*
(b) *Curves, and dominant rational maps between them;*
(c) *Finitely generated field extensions of k of transcendence degree 1, and field inclusions between them.*

This is a powerful tool for studying projective nonsingular curves, as we'll see soon.
Another consequence of the normalization construction is the following:

COROLLARY 7.4.6. *If C is a projective curve, X is any variety, and $\varphi : C \to X$ a non-constant morphism, then $\varphi(C)$ is a closed subset of X, which is a curve.*

If further C is nonsingular, then for every affine open subset U of $\varphi(C)$, we have that $\varphi^{-1}(U)$ is also affine, and the induced homomorphism $A(U) \to A(\varphi^{-1}(U))$ makes $A(\varphi^{-1}(U))$ into a finitely generated $A(U)$-module.

Note that in particular, if X itself is a curve, then φ is surjective. For the significance of the second part of the corollary statement, see Remark 7.4.9.

PROOF. Let D be the closure of $\varphi(C)$; then by Exercise 5.3.20, we have that D is a variety of dimension at most 1. Since φ is non-constant, we conclude that D is a curve, and we now restrict our attention to $\varphi : C \to D$, which we wish to show is surjective. By construction, φ maps C dominantly onto D, so it induces an injection $K(D) \hookrightarrow K(C)$, and since both function fields have transcendence degree 1, we conclude $K(C)$ is algebraic over $K(D)$. Since they are both finitely generated over k, $K(C)$ is finitely generated over $K(D)$, so is a finite extension. We can thus let $\widetilde{D}_C$ be the normalization of D in $K(C)$. Now, $\widetilde{D}_C$ is birational to C, so we have a rational map $\psi : \widetilde{D}_C \dashrightarrow C$ such that $\varphi \circ \psi$ is equal to the normalization map (considered as a rational map). But because $\widetilde{D}_C$ is nonsingular and C is projective, ψ extends to a morphism which must satisfy the same relation that $\varphi \circ \psi$ is equal to the normalization. But the normalization map is surjective, so we conclude that φ is likewise surjective.

Now, if C is nonsingular, we have that ψ is an isomorphism by Corollary 7.1.8, so since the second statement of the corollary is invariant under isomorphisms, it follows from the corresponding properties of normalization (Corollary 7.3.15). □

As an easy consequence, we deduce the following.

Corollary 7.4.7. *If $\varphi : C \to D$ is a nonconstant morphism of curves, then $\varphi(C)$ is an open subset of D.*

Proof. Since D has the cofinite topology, it is enough to show that $\varphi(C)$ contains an open subset of D. We may therefore restrict to the sets of nonsingular points of both C and D, so we reduce to the case that C and D are nonsingular. Now let $\bar{C}$ and $\bar{D}$ be the nonsingular projective compactification of C and D provided by Corollary 7.4.3. Then φ extends to a morphism $\bar{C} \to \bar{D}$, which we know is surjective by Corollary 7.4.6. But $\bar{C} \smallsetminus C$ consists of a finite set of points, so $\varphi(C)$ contains all but a finite set of points of $\bar{D}$, and in particular of D, so we conclude it is an open subset of D, as desired. $\square$

Remark 7.4.8. The first part of Corollary 7.4.6 generalizes to higher-dimensional varieties as the statement that the image of a morphism from a projective variety to a variety is always closed. Corollary 7.4.7 also generalizes, to the statement that the image of any dominant morphism of varieties contains an open subset of the target. The latter (in a slightly strengthened form) is known as Chevalley's theorem. We will prove both statements shortly, in Corollary 8.3.3 (ii) and Theorem 8.1.2 below.

Remark 7.4.9. The conclusion of the last part of Corollary 7.4.6 may appear technical, but as with the notion of normality, it has strong geometric consequences. In general, a morphism $\varphi : X \to Y$ of varieties is called **finite** if it has the property that for every affine open $V \subseteq Y$, we have $\varphi^{-1}(V)$ also affine, making $A(\varphi^{-1}(V))$ into a finitely generated $A(V)$-module. There are a number of equivalent characterizations of finiteness, but it implies in particular that φ has finite fibers, and that φ is closed. We will apply this notion to develop a criterion for imbedding nonsingular projective curves in other varieties in Corollary 9.4.3 below.

CHAPTER 8

Chevalley's theorem and complete varieties

In this chapter, we investigate the concept which plays the role of compactness for varieties – completeness. We prove that completeness can be characterized in terms of existence of extensions of morphisms from nonsingular curves, and conclude that projective varieties are complete. As a prelude to this, we also prove Chevalley's theorem on images of morphisms.

8.1. Chevalley's theorem

We have already seen that the image of a morphism of a variety need not be a subvariety (that is, it need not be a closed subset of an open subset). We recall the example:

EXAMPLE 8.1.1. Consider the morphism $\mathbb{A}^2 \to \mathbb{A}^2$ determined by $(x, y) \mapsto (x, xy)$. Its image is $\mathbb{A}^2 \smallsetminus Z(x) \cup \{(0, 0)\}$.

Chevalley's theorem asserts that this example is typical: the image of a morphism is always a finite union of subvarieties.

THEOREM 8.1.2 (Chevalley). *Let $\varphi : X \to Y$ be a morphism of prevarieties. Then $\varphi(X)$ can be written as a finite disjoint union of (not necessarily closed) subprevarieties of Y.*

Note that $\varphi(X)$ must be irreducible, so there is a unique subvariety of Y contained in $\varphi(X)$ which is dense in $\varphi(X)$; the other subvarieties of the theorem are all in its closure. The key statement to prove is:

PROPOSITION 8.1.3. *If $\varphi : X \to Y$ is a dominant morphism of prevarieties, then $\varphi(X)$ contains an open subset of Y.*

The first step is to show that any dominant morphism factors as a composition of two particular types of dominant morphisms, which will be easier to analyze.

LEMMA 8.1.4. *If $\varphi : X \to Y$ is a dominant morphism of affine varieties, and r is the transcendence degree of the induced field extension $K(X)/K(Y)$, then φ factors as a composition of dominant morphisms $X \to Y \times \mathbb{A}^r \to Y$, where the second morphism is the projection morphism.*

PROOF. Let $f_1, \dots, f_m$ be generators of $A(X)$ over $A(Y)$. Then the f_i also generate $K(X)$ over $K(Y)$, so we can reorder indices such that $f_1, \dots, f_r$ are algebraically independent over $K(Y)$. Let $R = A(Y)[f_1, \dots, f_r] \subseteq A(X)$. Since the f_i are algebraically independent over $K(Y)$ they are algebraically independent over $A(Y)$, so R is isomorphic to an r-variable polynomial ring, which is to say that $R \cong A(Y \times \mathbb{A}^r)$. Then the inclusions $A(Y) \hookrightarrow R \hookrightarrow A(X)$ induce the desired factorization. □

We can now prove that the image of a dominant morphism contains an open subset.

PROOF OF PROPOSITION 8.1.3. Let V be an affine open subset of Y, and U an affine open subset of X such that $\varphi(U) \subseteq V$. Then it clearly suffices to prove $\varphi(U)$ contains an open subset of V, so we have reduced to the affine case. Applying Lemma 8.1.4, it suffices to prove that the image of X in $Y \times \mathbb{A}^r$ contains an open subset, and that the projection morphism $Y \times \mathbb{A}^r \to Y$ is open.

For the first assertion, we have that the transcendence degrees of $K(X)$ and $K(Y \times \mathbb{A}^r)$ are equal, so the morphism makes $K(X)$ into an algebraic extension of $K(Y \times \mathbb{A}^r)$. Suppose $f_1, \dots, f_m$ generate $A(X)$ over $A(Y \times \mathbb{A}^r)$. Then each f_i is a root of some polynomial $g_i = \sum_j c_{i,j} t^j$ over $K(Y \times \mathbb{A}^r)$, which can assume to be monic. Let h be the product over all i, j of the denominators of $c_{i,j}$ (considering $K(Y \times \mathbb{A}^r)$ as the fraction field of $A(Y \times \mathbb{A}^r)$). We have the induced morphism of open subsets $X_h \to (Y \times \mathbb{A}^r)_h$, and we see that $A(X_h) = A(X)_h$ is still generated by the f_i over $A((Y \times \mathbb{A}^r)_h) = A(Y \times \mathbb{A}^r)_h$. But each f_i is integral over $A(Y \times \mathbb{A}_r)_h$ by construction, so by Proposition 7.3.6 and the ideal-variety correspondence, we have that X_h surjects onto $(Y \times \mathbb{A}^r)_h$, and thus the image of X contains the open subset $(Y \times \mathbb{A}^r)_h \subseteq Y \times \mathbb{A}^r$.

It remains to prove that for any $U \subseteq Y \times \mathbb{A}^r$ open, the image of U under the projection morphism $Y \times \mathbb{A}^r \to Y$ is open (we only need to prove it contains an open subset, but the stronger statement is no harder). Any such U is a union of open subsets of the form $(Y \times \mathbb{A}^r)_f$, for some nonzero $f \in A(Y \times \mathbb{A}^r) = A(Y)[t_1, \dots, t_r]$, so we may thus assume that U is of this form. Let $I \subseteq A(Y)$ be the ideal generated by the coefficients of f. We claim that the image of U is precisely $A(Y) \smallsetminus Z(I)$. Indeed, given $Q \in Y$, if we let Z_Q be the preimage of Q in $Y \times \mathbb{A}^r$, we have $Z_Q \cong \mathbb{A}^r$, and the inclusion $Z_Q \to Y \times \mathbb{A}^r$ is induced by the ring homomorphism $A(Y)[t_1, \dots, t_r] \to k[t_1, \dots, t_r]$ obtained by sending $g \in A(Y)$ to $g(Q)$. We thus see that $f|_{Z_Q}$ is obtained by evaluating the coefficients of f at Q, and so $Z_Q \cap U = Z_Q \smallsetminus Z(f)$ is non-empty if and only if f remains nonzero when its coefficients are evaluated at Q, which is precisely equivalent to the condition that $Q \notin Z(I)$. But Q is in the image of U if and only if $Z_Q \cap U \neq \emptyset$, so we conclude that the image of U is the complement of $Z(I)$, as desired. ☐

Chevalley's theorem then follows easily.

Proof of Theorem 8.1.2. Given $\varphi : X \to Y$, let $Z \subseteq Y$ be the closure of $\varphi(X)$; our proof is by induction on $\dim(Z)$. If $\dim(Z) = 0$, then φ is constant and there is nothing to show. Otherwise, assume $\dim(Z) = d > 0$, and we have the theorem already for dimensions smaller than d.

Now, we have a dominant morphism $X \to Z$, so let $U \subseteq Z$ be the maximal open subset of Z contained in $\varphi(X)$; this is nonempty by Proposition 8.1.3. Then let $Z' = Z \smallsetminus U$; this has dimension less than d, and is not necessary a prevariety, but can be write it as a finite union of prevarieties $Z_1, \dots, Z_m$. Similarly, $\varphi^{-1}(Z_i)$ is not necessarily a prevariety, but can be written as a finite union of prevarieties $X_{i,1}, \dots, X_{i,m_i} \subseteq X$. We have

$$\begin{aligned}\varphi(X) &= U \cup \varphi(\varphi^{-1}(Z_1)) \cup \cdots \cup \varphi(\varphi^{-1}(Z_m)) \\ &= U \cup \bigcup_{i,j} \varphi(X_{i,j}),\end{aligned}$$

and by the induction hypothesis each $\varphi(X_{i,j})$ is a finite union of subprevarieties of $Z_i \subseteq Y$, so we conclude the theorem. ☐

8.2. Completeness

We now apply our discussion of curves to give a definition for varieties which is analogous to the notion of compactness for topological spaces. We have the opposite problem that we had with the Hausdorff condition: every variety is compact in the Zariski topology, because its underlying topological space is Noetherian. However, the fix is the same as before: we give a rephrasing of the compactness condition in topology which will turn out to agree better with our intuition when we apply it to varieties.

Exercise 8.2.1. A topological space X is compact if and only if for every topological space Y, the projection map $X \times Y \to Y$ is a closed map.

Motivated by this, we define:

Definition 8.2.2. A variety X is **complete** if for all varieties Y, the projection morphism $X \times Y \to Y$ is a closed map.

Remark 8.2.3. One can apply the definition of completeness to prevarieties as well, but it is traditional to reserve the term "complete" for varieties. This is related to the French tradition that compactness should incorporate the Hausdorff condition as well. But see Exercise 8.2.10 below.

Example 8.2.4. $\mathbb{A}^n_k$ is not complete for $n > 0$. Indeed, if we take $Y = \mathbb{A}^1_k$ in the definition of complete, we have $\mathbb{A}^n_k \times \mathbb{A}^1_k \cong \mathbb{A}^{n+1}_k$, which contains the closed subvariety $x_1 x_{n+1} = 1$. The image of this under projection to $\mathbb{A}^1_k$ is the complement of the origin, which is not closed.

One reason for this definition is its relation to closed morphisms:

Proposition 8.2.5. *If X is complete, any closed subvariety of X is complete.*
If we also have Y an arbitrary variety, and $\varphi : X \to Y$ a morphism, then φ is closed.

Proof. The first assertion is immediate from the definition, since if Z is closed in X, we have $Z \times Y$ closed in $X \times Y$ for any Y.

For the second assertion, let $\Gamma = \{(x, \varphi(x) : x \in X\} \subseteq X \times Y$ be the graph of φ. We can express Γ as the preimage of the diagonal $\Delta(Y) \subseteq Y \times Y$ under the morphism $\varphi \times \mathrm{id} : X \times Y \to Y \times Y$. Since Y is a variety, $\Delta(Y)$ is closed, so Γ is closed. But $\varphi(X)$ is precisely the image of Γ under the projection $X \times Y \to Y$, so we conclude from the completeness hypothesis on X that $\varphi(X)$ is closed. $\square$

Remark 8.2.6. This is a natural property for complete varieties to have, since a continuous map from a compact topological space to a Hausdorff space is closed. In fact, this property characterizes complete varieties – Nagata proved that every variety can be realized as an open subset of a complete variety, so in particular if X is not complete, the inclusion as an open subset of a complete variety is not a closed mapping. However, the proof of Nagata's theorem is beyond the scope of this book.

A key property of complete varieties is that (in stark contrast to the quasiaffine case) their global regular functions are all constant.

Proposition 8.2.7. *Let X be a complete variety. Then every regular function on X is constant; that is, $\mathscr{O}(X) = k$.*

Proof. A regular function f on X yields a morphism $X \to \mathbb{A}^1_k$. We can compose with the inclusion $\mathbb{A}^1_k \subseteq \mathbb{P}^1_k$ to obtain a morphism $\varphi : X \to \mathbb{P}^1_k$. Now, X is complete and $\mathbb{P}^1_k$ is a variety, so by Proposition 8.2.5, we have $\varphi(X)$ closed in $\mathbb{P}^1_k$. On the other hand, X is irreducible, and the continuous image of an irreducible space is irreducible, so $\varphi(X)$ is closed and irreducible in $\mathbb{P}^1_k$. The only closed irreducible subsets of $\mathbb{P}^1_k$ are all of $\mathbb{P}^1_k$ or points, and since $\varphi(X) \subseteq \mathbb{A}^1_k$, we cannot have $\varphi(X) = \mathbb{P}^1_k$, so $\varphi(X)$ is a point, and the original regular function was constant, as desired. $\square$

Example 8.2.8. We now see that if X is both affine and complete, then it must be a single point, since the only affine variety with $A(X) \cong k$ is the point.

More generally, we see that we cannot imbed any positive-dimensional complete variety into affine space. In fact, we have the following.

Corollary 8.2.9. *If X is complete, and Y is affine, then any morphism $X \to Y$ is constant.*

Proof. Imbed $Y \subseteq \mathbb{A}^n_k$, a morphism $X \to Y$ induces a morphism $X \to \mathbb{A}^n_k$, which we know is determined by an n-tuple of regular functions. But these are all constant by Proposition 8.2.7, so the morphism is constant, as claimed. $\square$

EXERCISE 8.2.10. One could define a prevariety X to be **universally closed** if for all prevarieties Y, the projection map $X \times Y \to Y$ is closed. Thus, if X is universally closed and is also a variety, then X is complete. Show that conversely, if X is a complete variety, then X is universally closed. Show the stronger statement that a prevariety X is universally closed if and only if $X \times \mathbb{A}_k^n \to \mathbb{A}_k^n$ is closed for all n.

8.3. A limit-based criterion

We wish to give a more intuitive necessary and sufficient criterion for completeness. Because the ideas are closely related, we will also give a more intuitive criterion for a prevariety to be a variety. In the informal language of limits we discussed in Remark 7.1.9, we will show that a prevariety is a variety if and only if limits are unique when they exist, and that a variety is complete if and only if limits always exist. In a related but more general form, this theorem gives what are typically called valuative criteria.

THEOREM 8.3.1. *A prevariety X is a variety if and only if for all nonsingular curves C, and points $P \in C$, and morphisms $\varphi : C \smallsetminus \{P\} \to X$, there is at most one extension of φ to a morphism $C \to X$.*

A variety X is complete if and only if for all nonsingular curves C, and points $P \in C$, and morphisms $\varphi : C \smallsetminus \{P\} \to X$, there exists a (necessarily unique) extension of φ to a morphism $C \to X$.

Before giving the proof, we observe some consequences. We will immediately conclude from Theorems 8.3.1 and 7.1.5:

COROLLARY 8.3.2. *Any projective variety is complete.*

From Propositions 8.2.5 and 8.2.7, and Corollary 8.2.9, we then conclude:

COROLLARY 8.3.3. *Let X be a projective variety. Then:*
(i) *$\mathscr{O}(X) = k$;*
(ii) *any morphism from X to an arbitrary variety is closed;*
(iii) *any morphism from X to an affine variety is constant.*

In order to prove Theorem 8.3.1, an important lemma is the following:

LEMMA 8.3.4. *Suppose $\varphi : X \to Y$ is a morphism of prevarieties, and $Q \in Y$ is in the closure of $\varphi(X)$. Then there exists a nonsingular curve C and $P \in C$, and morphisms $\widetilde{\psi} : C \smallsetminus \{P\} \to X$ and $\psi : C \to Y$ such that $\varphi \circ \widetilde{\psi} = \psi|_{C \smallsetminus \{P\}}$, and and $\psi(P) = Q$.*

An intermediate lemma is the following.

LEMMA 8.3.5. *Suppose X is a prevariety, U a nonempty open subset, and $P \in X \smallsetminus U$. Then there exists a subprevariety Z of X which is a curve, such that $P \in Z$, and $U \cap Z \neq \emptyset$.*

PROOF. It suffices to produce such a Z inside any open neighborhood of P in X, so let $V \subseteq X$ be an affine open neighborhood of Q. We prove the statement by induction on $\dim X$; if $\dim X = 1$, we simply let $Z = X$. For $\dim X > 1$, let $\mathfrak{m}_P$ be the maximal ideal of $A(V)$ corresponding to P, and let $I \subseteq A(V)$ be the radical ideal with $V \smallsetminus U = Z(I)$. Then by Exercise 4.3.8, there exists f such that $f(P) = 0$, but f does not vanish uniformly on any irreducible component of $Z(I)$ having codimension 1. Now, let X' be an irreducible component of $Z(f)$ which contains P. Then $\dim X' = \dim X - 1$, and since $Z(f)$ doesn't contain any component of $Z(I)$ of codimension 1, we find that X' cannot be contained in $Z(I)$, so $X' \cap U \neq \emptyset$. Replacing X by X' and U by $X' \cap U$, we apply the induction hypothesis to find the desired curve. $\square$

We can now prove the first lemma.

Proof of Lemma 8.3.4. Our first claim is that there is a subprevariety $D \subseteq Y$ which is a curve containing Q, and such that $D \cap \varphi(X)$ is dense in D. But by Proposition 8.1.3, if Y' is the closure of $\varphi(X)$ in Y, we know that $\varphi(X)$ contains an open subset of Y', so this follows immediately from Lemma 8.3.5. Now, let $Z \subseteq X$ be the preimage of D under φ. Since $D \cap \varphi(X)$ is dense in D, some irreducible component X' of Z maps dominantly to D. Choose $Q' \in D \cap \varphi(X')$; then $\varphi^{-1}(Q') \cap X'$ is closed in X', and cannot be all of X'. Again using Lemma 8.3.5, there is a curve C' in X' which meets $\varphi^{-1}(Q') \cap X'$ but is not contained in it. We then see that C' maps dominantly to D, since its image must be irreducible and strictly contains Q'. Let $\widetilde{C}$ be the normalization of D inside $K(C')$, and let $P \in \widetilde{C}$ be a point mapping to Q. By construction, $K(\widetilde{C}) \cong K(C')$, so we get a birational map $\widetilde{C} \dashrightarrow C'$ commutes with the given morphisms to D. Let $U \subseteq \widetilde{C}$ be the open subset on which the rational map is defined. We can set $C = U \cup \{P\}$, which is still an open subset of $\widetilde{C}$, and we obtain morphisms $C \to D \subseteq Y$ and $C \smallsetminus \{P\} \to C' \subseteq X' \subseteq X$ satisfying the desired conditions. ☐

Before proving the theorem, we give one more lemma.

Lemma 8.3.6. *Let $\varphi : C \to D$ be a birational morphism of curves, with D nonsingular. Then φ is an isomorphism of C onto the open subset $\varphi(C)$ of D.*

Proof. Let $\nu : \widetilde{C} \to C$ be the normalization, and let $\bar{C}$ and $\bar{D}$ be the nonsingular projective curves having $\widetilde{C}$ and D as open subsets, respectively. Then $\varphi \circ \nu$ induces a birational map $\bar{C} \to \bar{D}$, which is necessarily an isomorphism. But then both $\widetilde{C}$ and D are identified as open subsets of a given curve, compatibly with the morphism $\varphi \circ \nu$, so we conclude that $\varphi \circ \nu$ must map $\widetilde{C}$ isomorphically onto an open subset of D. By the surjectivity of ν, this open subset is $\varphi(C)$, and then the morphism $\varphi(C) \to \widetilde{C} \to C$ show that φ is an isomorphism onto $\varphi(C)$, as desired. ☐

We now prove the promised theorem.

Proof of Theorem 8.3.1. For the first statement, we already know that if X is a variety, then the stated condition holds. Conversely, suppose the condition holds, and consider the diagonal morphism $\Delta : X \to X \times X$. Let $Q \in X \times X$ be in the closure of $\Delta(X)$. Then by Lemma 8.3.4, there is a nonsingular curve C and a point $P \in C$, with morphisms $\widetilde{\psi} : C \smallsetminus \{P\} \to X$ and $\psi : C \to X \times X$ such that $\psi(P) = Q$, and $\psi = \Delta \circ \widetilde{\psi}$ on $C \smallsetminus \{P\}$. We then get two extensions of $\widetilde{\psi}$ to all of C by composing ψ with the projection morphisms p_1, p_2. By hypothesis, these extensions are unique, so we conclude that $p_1(\psi(P)) = p_2(\psi(P))$, so $Q = \psi(P) \in \Delta(X)$, and $\Delta(X)$ is closed.

Now, suppose X is a complete variety. Given C, the point $P \in C$, and $\varphi : C \smallsetminus \{P\} \to X$, consider the product $X \times C$. Let Z be the closure of $\{(\varphi(P), P) : P \in C\} \subseteq X \times C$. Then we have dominant morphisms $C \smallsetminus \{P\} \to Z \to C$, and the image of Z is closed in C by hypothesis, so we must have $Z \to C$ surjective. By Lemma 8.3.6, we conclude that $Z \to C$ is an isomorphism. Inverting the isomorphism, it follows that we have a morphism $C \to Z$ extending $C \smallsetminus \{P\} \to Z$, and taking the first projection we get the desired extension of φ to all of C.

Conversely, suppose X satisfying the stated condition. Given any variety Y, let Z be a closed subset of $X \times Y$. Let $Q \in Y$ be in the closure of the image of Z under the projection morphism p_2. By Lemma 8.3.4, there is a nonsingular curve C and a point $P \in C$, with morphisms $\widetilde{\psi} : C \smallsetminus \{P\} \to Z$ and $\psi : C \to Y$ such that $\psi(P) = Q$, and $\psi = p_2 \circ \widetilde{\psi}$ on $C \smallsetminus \{P\}$. By hypothesis, we can extend $p_1 \circ \widetilde{\psi}$ to a morphism $\psi' : C \to X$, and we see that if we take the product morphism $\psi' \times \psi : C \to X \times Y$, it extends $\widetilde{\psi}$, and must therefore have image contained in Z, since Z is closed. Moreover, by definition we have that $Q = p_2((\psi' \times \psi)(P))$, so $Q \in p_2(Z)$, and we conclude $p_2(Z)$ is closed. ☐

Remark 8.3.7. There are direct, algebraic proofs of Corollary 8.3.2. See for instance Theorem 2 of Chapter I, §5.2 of [**Sha94a**]. However, the point of our approach is to show that if one builds up enough foundational tools, one can start to prove interesting results more geometrically, without resorting to going back to definitions and using algebra to prove each result.

Exercise 8.3.8. In this exercise, we give a proof of Chow's Lemma. It is clear that every complete variety is birational to some projective variety. However, Chow's Lemma asserts that a much stronger statement is true: given a complete variety X, there exists a projective variety X' together with a birational morphism $X' \to X$ (which is necessarily surjective, by Corollary 8.3.3 (ii)).

Let $\{U_i\}$ be an affine open cover of X, and let Y_i be the closures of the U_i in projective space. Let U be the intersection of the U_i, and

$$\varphi : U \to X \times Y_1 \times \cdots \times Y_n$$

the morphism induced by the inclusions of U into X and the Y_i. Let X' be the closure of $\varphi(U)$. Let $p_1 : X' \to X$ be the morphism induced by the first projection, and $p_2 : X' \to Y_1 \times \cdots \times Y_n$ be the morphism induced by projection to the remaining factors.

(a) Show that p_1 gives an isomorphism $p_1^{-1}(U) \to U$. Hint: first prove that $\varphi(U)$ is an open subset of X'.

(b) Show that p_2 induces an isomorphism of X' onto a closed subvariety of $Y_1 \times \cdots \times Y_n$. Hint: first prove that

$$X' \cap X \times Y_1 \times \cdots \times U_i \times \cdots \times Y_n = X' \cap U_i \times Y_1 \times \cdots \times Y_n,$$

by considering the projections to X and to Y_i for each.

(c) Conclude Chow's lemma.

8.4. Irreducibility of polynomials in families

In this section, we give an application of the ideas we have discussed, studying how irreducibility behaves in families of polynomials. We further apply this to prove a theorem on existence of chains of curves connecting any two points of a prevariety.

The key input involves the fact that projective varieties are complete, from which we are able to conclude the following.

Lemma 8.4.1. *Given integers $n, m, n_1, \ldots, n_m$, suppose a morphism*

$$\varphi : \prod_i \mathbb{A}_k^{n_i} \to \mathbb{A}_k^n$$

is given by nonconstant polynomials

$$f_1, \ldots, f_n \in k[x_{i,j}]_{1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n_i},$$

such that:

(i) *there exist $d_1, \ldots, d_m$ such that each f_i is homogeneous of degree d_j in the variables $x_{j,1}, \ldots, x_{j,n_j}$;*

(ii) *we have*

$$\varphi^{-1}(\{0\}) \subseteq \bigcup_{i=1}^{m} \mathbb{A}_k^{n_1} \times \cdots \times \mathbb{A}_k^{n_{i-1}} \times \{0\} \times \mathbb{A}_k^{n_{i+1}} \times \cdots \times \mathbb{A}_k^{n_m}.$$

Then the image of φ is closed.

PROOF. The hypotheses imply that φ induces a morphism $\bar{\varphi} : \prod_i \mathbb{P}_k^{n_i-1} \to \mathbb{P}_k^{n-1}$ such that $p_2 \circ \varphi = p_2$, which has closed image by Proposition 8.2.5, together with Corollary 8.3.2. Again using homogeneity, the image of φ contains $\{0\}$. But using the canonical morphisms $\pi_r : \mathbb{A}_k^r \smallsetminus \{0\} \to \mathbb{P}_k^{r-1}$, we see that the image of φ on the complement of $\{0\}$ is precisely the preimage under π_n of the image of $\bar{\varphi}$. Thus, the image of φ is closed after restriction to the complement of $\{0\}$, and it contains $\{0\}$, so it must be closed. $\square$

We now describe our application to irreducibility of polynomials in families. A natural situation to consider a family of polynomials in algebraic geometry is to give ourselves an affine variety X, and an element $f \in A(X)[x_1, \dots, x_n]$. Then for every point P of X, we get a polynomial $f(P) \in k[x_1, \dots, x_n]$, and the condition that the coefficients of f be regular functions on X ensures that the family reflects the geometry of X. In this setting, we have the following basic result.

COROLLARY 8.4.2. *Let X be an affine variety, and $f \in A(X)[x_1, \dots, x_n]$ a polynomial of degree d. Let*

$$Z = \{P \in X : f(P) \text{ is reducible or } \deg f(P) < d.\}.$$

Then Z is a closed subset.

In the above definition of Z, if $f(P)$ is the zero polynomial, we consider it to have degree strictly less than d.

PROOF. Given any $d' \geqslant 0$, let $\mathbb{A}_{n,d'}$ denote the affine space whose coordinates are identified with the coefficients of a general polynomial of degree d' in the variables $x_1, \dots, x_n$. Thus, the points of $\mathbb{A}_{n,d'}$ correspond to elements of $k[x_1, \dots, x_n]$ having degree at most d'. More precisely, we have a polynomial $f_{n,d'} \in A(\mathbb{A}_{n,d'})[x_1, \dots, x_n]$ of degree d', with the coefficient of each monomial in the x_i's being the corresponding coordinate of $\mathbb{A}_{n,d'}$, and having the property that for every $g \in k[x_1, \dots, x_n]$ of degree at most d', there is a unique point P of $\mathbb{A}_{n,d'}$ such that $f_{n,d'}(P) = g$.

We first consider the "universal" case that $X = \mathbb{A}_{n,d}$, and $f = f_{n,d'}$. In this case, write $Z_{\text{univ}} \subseteq \mathbb{A}_{n,d}$ for the Z of the statement. For every d_1, d_2 positive with $d_1 + d_2 = d$, we have a morphism

$$\mu_{d_1,d_2,n} : \mathbb{A}_{n,d_1} \times \mathbb{A}_{n,d_2} \to \mathbb{A}_{n,d}$$

corresponding to polynomial multiplication. This morphism is given by polynomials which are homogeneous (indeed, linear) in the variables coming from the first and second factors. Moreover, because a product of two polynomials is zero if and only if one of the polynomials is zero, we see that the hypotheses of Lemma 8.4.1 are satisfied, so the image of $\mu_{d_1,d_2,n}$ is closed. On the other hand, it is clear by construction that Z_{univ} is the union of the $\mu_{d_1,d_2,n}$ as d_1, d_2 vary, so we conclude that Z_{univ} is closed.

In the case that X and f are arbitrary, we observe that the coefficients of f determine a morphism $X \to \mathbb{A}_{n,d}$ with the property that $Z \subseteq X$ is exactly the preimage of $Z_{\text{univ}} \subseteq \mathbb{A}_{n,d}$, so we conclude that Z is also closed, as desired. $\square$

REMARK 8.4.3. The argument for Corollary 8.4.2 illustrates a common technique in algebraic geometry: reduction to a suitable universal situation. Frequently, a general statement can be reduced to proving the same statement in a single, universal instance, and the universal case, being more explicit, is more amenable to direct analysis. As in the above argument, usually once the universal situation is correctly described, the processing of reducing to that case is more or less a formality.

EXAMPLE 8.4.4. The statement of Corollary 8.4.2 would be false if we simply defined Z to be the set of points P such that $f(P)$ is reducible. For instance, we could take $X = \mathbb{A}_k^1$ with coordinate t, and $f = tx^2 + x$. This is reducible for all $t \neq 0$, but irreducible for $t = 0$.

We now refine Corollary 8.4.2 to the following theorem.

THEOREM 8.4.5. *Let X be an affine variety, and $f \in A(X)[x_1, \dots, x_n]$ a polynomial of degree d. Let*

$$Z = \{P \in X : f(P) \text{ is reducible or } \deg f(P) < d.\}.$$

Then Z is a closed subset.

Moreover, if f is irreducible over $\overline{K(X)}$, then $Z \neq X$.

The main additional ingredient is a technique which we have not used elsewhere: extension of base field. Given an algebraically closed field k' extending k, and an affine algebraic set X over k, then X is defined by polynomials with coefficients in k, so the extension k'/k canonically gives us an affine variety X' over k' defined by the same polynomials. Given also Y another affine algebraic set over k, and $\varphi : X \to Y$ a morphism, then we know that φ is likewise given by polynomials with coefficients in k, so we also get a canonical morphism $\varphi' : X' \to Y'$ defined by the same polynomials. Similarly, if $Z \subseteq Y$ is a closed algebraic subset, we get a closed algebraic subset $Z' \subseteq Y'$ defined by the same polynomials as Z. The key fact for us is that if $\varphi(X) \subseteq Z$, then also $\varphi'(X') \subseteq Z'$. Indeed, suppose Z is defined by polynomials $f_1, \dots, f_n$. Then to say that $\varphi(X) \subseteq Z$ is equivalent to saying that $f_i \circ \varphi = 0$ in $A(X)$ for $i = 1, \dots, n$, or $f_i \circ \varphi \in I(X)$. But $I(X) \subseteq I(X')$ by definition, so $f_i \circ \varphi \in I(X')$, and since φ' is defined by the same polynomials as φ, we conclude that $\varphi(X') \subseteq Z'$, as desired.

PROOF OF THEOREM 8.4.5. Following the above discussion, set $k' = \overline{K(X)}$. Extending from k to k', and following the setup of the proof of Corollary 8.4.2, we have a morphism $X' \to \mathbb{A}'_{n,d}$ and a closed algebraic subset $Z'_{\text{univ}} \subseteq \mathbb{A}'_{n,d}$. Now, the polynomials defining X have a canonical solution over $A(X)$ just by the definition of $A(X)$, using the images in $A(X)$ of the coordinate functions themselves. Using the inclusions $A(X) \subseteq K(X) \subseteq k'$, we find that X' has a canonical point. By construction, the image of this point in $\mathbb{A}'_{n,d}$ corresponds precisely to considering f over k'. Since f is assumed irreducible over k', the image of this point is not in Z'_{univ}, so we conclude that the image of X' is not contained in Z'_{univ}, and hence by the above discussion that the image of X is not contained in Z_{univ}, meaning that $Z \neq X$, as desired. $\square$

EXAMPLE 8.4.6. In the statement of Theorem 8.4.5, it would not be enough to assume that f is irreducible in $A(X)[x_1, \dots, x_n]$. Indeed, consider the case $n = 1$: since $K(X)$ is not algebraically closed, there are irreducible nonlinear polynomials in $A(X)[x_1, \dots, x_n]$. But when evaluated at P, none of these can remain irreducible, since k is algebraically closed.

REMARK 8.4.7. Theorem 8.4.5, while classical, is best understood in the scheme setting. Indeed, if one modifies the definition of Z appropriately (specifically, by always considering irreducibility over the appropriate algebraic closures), then the argument of Corollary 8.4.2 shows that Z is still closed in the corresponding scheme. The condition that f remains irreducible over $\overline{K(X)}$ says exactly that the generic point of the scheme corresponding to X is not in Z, so it follows immediately that the complement of Z is a nonempty open subset.

We next have the following generalization of Theorem 3.3.14.

THEOREM 8.4.8. *If X is a variety of dimension d, then given any n with $0 \leqslant n \leqslant d$, there exists an affine variety Y of dimension n, and an irreducible polynomial $f \in A(Y)[x_1, \dots, x_{d+1-n}]$ such that $Z(f)$ is birational to X, and f remains irreducible over $\overline{K(Y)}$.*

Note that Theorem 3.3.14 is the special case $n = 0$. We will apply Theorem 8.4.8 in the case $n = 1$ in order to make arguments regarding general varieties which induct on dimension.

Proof. As in the proof of Theorem 3.3.14, there exist algebraically independent $y_1, \dots, y_d \in K(X)$ such that $K(X)$ is a finite, separable extension of $k(y_1, \dots, y_d)$. Let K be the algebraic closure of $k(y_1, \dots, y_n)$ inside $K(X)$ – that is, the set of elements of $K(X)$ algebraic over $k(y_1, \dots, y_n)$. Then K is necessarily separable over $k(y_1, \dots, y_n)$, and we claim that it is a finite extension, so in particular, finitely generated over k. Observe that the primitive element theorem implies that a separable algebraic extension which is not finite contains elements of arbitrarily high degree. But one checks that if $\alpha \in K(X)$ is algebraic over $k(y_1, \dots, y_n)$, then its minimal polynomial over $k(y_1, \dots, y_d)$ is the same as its minimal polynomial over $k(y_1, \dots, y_n)$, and because $K(X)$ is a finite extension of $k(y_1, \dots, y_d)$, the elements of $K(X)$ have bounded degree over $k(y_1, \dots, y_d)$. We thus conclude that the elements of K have bounded degree over $k(y_1, \dots, y_n)$, and that K is finite over $k(y_1, \dots, y_n)$, as desired.

Consequently, we can write $K = K(Y)$ for some affine variety Y of dimension n. By construction, we can realize $K(X)$ as a finite separable extension of the purely transcendental extension $K(Y)(y_{n+1}, \dots, y_d)$ of $K(Y)$, so using the primitive element theorem again, we have $K(X) \cong K(Y)(y_{n+1}, \dots, y_d)[t]/f$ for some irreducible $f \in K(Y)(y_{n+1}, \dots, y_d)[t]$. We can clear denominators to assume that $f \in A(Y)[y_{n+1}, \dots, y_d, t]$ (still irreducible), and we have $Z(f)$ birational to X by construction. Finally, the fact that f remains irreducible over $\overline{K(Y)}$ is a consequence of the fact that $K(X)$ is separably generated over $K(Y)$, and $K(Y)$ is algebraically closed in $K(X)$; see Exercise A.1.2 of [**Eis95**] for a more general statement. □

Putting together Theorems 8.4.5 and 8.4.8, we are able to conclude the following.

Corollary 8.4.9. *Let X be a variety of dimension d. Then for any n with $0 < n < d$, there exists a variety Y of dimension n, a nonempty open subset $U \subseteq X$, and a dominant morphism $\varphi : U \to Y$ such that all nonempty fibers of φ are irreducible of dimension $d - n$.*

Note that Chevalley's theorem implies that the image of X in Y contains a nonempty open subset of Y, so if we restrict to this subset we can also require that the fibers of φ are all nonempty.

Proof. First, let Y' and $f \in A(Y')[x_1, \dots, x_{d+1-n}$ be as given to us by Theorem 8.4.8. We are given that f remains irreducible over $\overline{K(Y')}$, so according to Theorem 8.4.5, there is a nonempty open subset V of Y' such that $f(P)$ is irreducible of the same degree for all $P \in V$. If we let Y be an affine open subset of Y' contained in V, we can consider f to be in $A(Y)[x_1, \dots, x_{d+1-n}]$, and we have that if $Z = Z(f) \subseteq Y \times \mathbb{A}^{d+1-n}$, the fibers of Z over Y are all irreducible of dimension $d - n$. But by construction, Z is birational to X, so letting $U \subseteq X$ be an open subset which is isomorphic to an open subset of Z gives the desired morphism to Y. □

Finally, we can conclude that any two points on a (pre)variety can be connected by a chain of curves.

Theorem 8.4.10. *Let X be a prevariety of positive dimension, and $P, P' \in X$. Then there exist closed one-dimensional subprevarieties $Z_1, \dots, Z_n \subseteq X$ such that $P \in Z_1$, $P' \in Z_n$, and $Z_i \cap Z_{i+1} \neq \emptyset$ for $i = 1, \dots, n-1$.*

Proof. The proof is by induction on the dimension d of X, with the case $d = 1$ being trivial. By Lemma 8.3.5, we are free to restrict to open subsets of X, so by Corollary 8.4.9 in the case $n = 1$, we may assume we have a one-dimensional variety Y and a morphism $\varphi : X \to Y$ whose nonempty fibers are irreducible of dimension $d - 1$. Now, given any $P \in X$, applying Lemma 8.3.5 again to the complement of $\varphi^{-1}(\varphi(P))$, we find a curve $Z \subseteq X$ not contained in $\varphi^{-1}(\varphi(P))$. Then φ must map Z dominantly to Y, so $\varphi(Z) \subseteq Y$ is open, and replacing X with $\varphi^{-1}(\varphi(Z))$, we may assume that Z surjects onto Y. Then Z connects the fibers $\varphi^{-1}(\varphi(P))$ to every other fiber of φ, and applying the induction hypothesis to the fibers (and taking the closure of Z), we conclude that

P can be connected to any other point of X by a chain of closed one-dimensional subprevarieties, as desired. $\square$

CHAPTER 9

Divisors on nonsingular curves

We now begin a closer study of the behavior of projective nonsingular curves, and morphisms between them, as well as to projective space. To this end, we introduce and study the concept of divisors.

9.1. Morphisms of curves

Suppose $\varphi : X \to Y$ is a nonconstant morphism of curves. Then we know that φ is dominant. The induced field extension $K(Y) \hookrightarrow K(X)$ must be algebraic, since $K(X)$ and $K(Y)$ have the same transcendence degree, and indeed it must be finite, since $K(X)$ is finitely generated over k.

Definition 9.1.1. The **degree** of a morphism $\varphi : X \to Y$ of curves is 0 if φ is constant, and is $[K(X) : K(Y)]$ otherwise.

In the case that X, Y are projective, nonsingular curves and φ is nonconstant, we already know that φ is necessarily surjective, but we will prove (more accurately, sketch a proof of) a much stronger result.

Definition 9.1.2. Suppose $\varphi : X \to Y$ is a nonconstant morphism of nonsingular curves, and $P \in X$ any point. The **ramification index** e_P of φ at P is defined as follows: let $t \in \mathscr{O}_{f(P),Y}$ be a local coordinate at P (equivalently, an element with $\mathrm{ord}_{f(P)}(t) = 1$); then $e_P = \mathrm{ord}_P(\varphi_P^*(t))$.

Observe that e_P is well defined, since φ_P^* is injective, and a different choice of t will differ by multiplication by a unit, which doesn't affect $\mathrm{ord}_P(\varphi_P^*(t))$. Also, $e_P \geqslant 1$ always, since $\varphi_P^* t$ must vanish at P.

Definition 9.1.3. Suppose $\varphi : X \to Y$ is a morphism of nonsingular curves, and $P \in X$ any point. Then P is a **ramification point** of φ if $e_P \geqslant 2$. In this case, we say $\varphi(P)$ is a **branch point** of φ. If $e_P = 1$, we say that φ is **unramified** at P.

Remark 9.1.4. Conceptually (and in fact precisely, when one is working over $\mathbb{C}$), a ramification point is a critical point of φ (i.e., a point where the derivative of φ vanishes), and a branch point is a critical value. We will discuss a closely related version of this statement after we have introduced differential forms.

The fundamental result in the case that X, Y are projective is:

Theorem 9.1.5. *Let $\varphi : X \to Y$ be a nonconstant morphism of projective, nonsingular curves, having degree d, and let $Q \in Y$ any point. Then*

$$\sum_{P \in \varphi^{-1}(Q)} e_P = d.$$

Sketch of proof. We use the projectivity hypothesis only to apply Corollary 7.4.6, concluding that if $V \subseteq Y$ is an affine open neighborhood of Q, then $U := \varphi^{-1}(V)$ is also affine, and furthermore $A(U)$ is a finitely generated $A(V)$-module.

The desired result now follows from a standard result in algebra on the behavior of extensions of Dedekind domains. In our case, if we use the notation

$$\mathscr{O}_{Q,X} := \bigcap_{P\in\varphi^{-1}(Q)} \mathscr{O}_{P,X},$$

the main idea is to show that by the classification of finitely generated modules over principal ideal domains, we have $\mathscr{O}_{Q,X}$ a free module over $\mathscr{O}_{Q,Y}$ of rank d, and then (if $t \in \mathscr{O}_{Q,Y}$ is a local coordinate) to use the Chinese remainder theorem to relate $\mathscr{O}_{Q,X}/(\varphi^*t)$ to the various $\mathscr{O}_{P,X}/(\varphi^*t)$, each of which one can show is e_P-dimensional. $\square$

The geometric intuition behind the theorem is that at most points, the morphism φ is a $d:1$ cover, but that at certain points (the ramification points), some of the d sheets come together.

Example 9.1.6. Suppose $X = Y = \mathbb{P}^1$, and φ is given by $(X_0, X_1) \mapsto (X_0^d, X_1^d)$. Away from $X_0 = 0$, we can normalize so that $X_0 = 1$, and the morphism is the morphism $\mathbb{A}^1 \to \mathbb{A}^1$ given by $x \mapsto x^d$. We first assume that we are not in the situation that $\operatorname{char} k = p$ and $p|d$. If $c \neq 0$, the preimage of $x = c$ consists of d distinct points, so we see that each of these d points must be unramified. However, over $c = 0$, we have only a single preimage, so the ramification index at 0 must be d. The situation is symmetric for $x_1 \neq 0$, so we find that the ramification points are $(1, 0)$ and $(0, 1)$, each with ramification index d, and all the other points are unramified.

Now, suppose that $\operatorname{char} k = p$ and $p|d$. Write $d = p^r d'$, with p not dividing d'. In this case, we see that all points are ramified; $(1, 0)$ and $(0, 1)$ still have ramification index d, while the rest all have ramification index p^r.

Remark 9.1.7. Theorem 9.1.5 has a parallel result in classical algebraic number theory, describing how prime ideals in a ring of integers factor when one extends to a larger ring of integers. It is one of the appealing aspects of Grothendieck's theory of schemes that it allows one to phrase a single theorem which simultaneously encompasses both results.

Exercise 9.1.8. Show that a nonconstant morphism $\varphi : \mathbb{P}^1 \to \mathbb{P}^1$ is ramified at all points of $\mathbb{P}^1$ if and only if it factors through the Frobenius morphism.

Exercise 9.1.9. Let X, Y be nonsingular curves, and $\varphi : X \to Y$ a nonconstant morphism. For $P \in X$, show that φ is unramified at P if and only if the induced map $T_P(X) \to T_{\varphi(P)}(Y)$ is injective.

Our next goal is to study the behavior of such morphisms in more detail. In order to do so, we will have to introduce the concepts of divisors and differential forms.

9.2. Divisors on curves

An important topic in classical algebraic geometry is the study of divisors. They play a crucial role in understanding morphisms to projective space, and will also be important for us in our study of morphisms between curves. We will restrict our treatment to the case of nonsingular curves, although most of the basic definitions generalize rather easily to the case of higher-dimensional nonsingular varieties (with points being replaced by closed subvarieties of codimension 1). We will assume throughout this section that X is a nonsingular curve.

Definition 9.2.1. A **divisor** D on X is a finite formal sum $\sum_i c_i[P_i]$, where $c_i \in \mathbb{Z}$, and each P_i is a point of X. Given also $D' = \sum_i c_i'[P_i]$, we write $D \geqslant D'$ if $c_i \geqslant c_i'$ for each i. We say D is **effective** if $D \geqslant 0$. The **degree** $\deg D$ of D is defined to be $\sum_i c_i$.

We can define pullbacks of divisors under morphisms as follows:

Definition 9.2.2. If X, Y are nonsingular curves, and $\varphi : X \to Y$ is a nonconstant morphism, and $D = \sum c_i[Q_i]$ a divisor on Y, we define the **pullback** of D under φ, denoted $\varphi^*(D)$, to be the divisor $\sum_i \sum_{P \in \varphi^{-1}(Q_i)} e_P c_i [P]$ on X.

We then have the following corollary, which is an immediate consequence of Theorem 9.1.5.

Corollary 9.2.3. *Let $\varphi : X \to Y$ be a morphism of projective nonsingular curves, of degree d. Then for any divisor D on Y, we have*

$$\deg \varphi^*(D) = d \deg D.$$

Divisors are closely related to the study of rational functions.

Definition 9.2.4. Given $f \in K(X)^*$, the associated divisor $D(f)$ is $\sum_{P \in X} \operatorname{ord}_P(f)[P]$. A divisor D is **principal** if $D = D(f)$ for some $f \in K(X)^*$.

Note that $D(f)$ is indeed a divisor: f is regular away from a finite number of points, and where f is regular, it only vanishes at a finite numbers of points.

Remark 9.2.5. The terminology of principal divisor is suggestive, and indeed there is a close relationship between principal divisors and principal ideals. One can develop this connection in the context of affine varieties, but the most satisfying treatment, involving a single definition encompassing both principal ideals of rings of integers and principal divisors on projective curves, requires the theory of schemes.

If we stick to projective curves, we find that principal divisors are quite well behaved. In particular:

Proposition 9.2.6. *A principal divisor on a projective nonsingular curve has degree* 0.

Proof. Let $D(f)$ be a principal divisor on the projective nonsingular curve X. If f is constant, then $D(f) = 0$, so the degree is visibly 0. If f is nonconstant, we have seen that it defines a dominant rational map to $\mathbb{A}^1$, which we may extend to a morphism $f : X \to \mathbb{P}^1$. We claim that $D(f) = f^*([0] - [\infty])$. But this is clear: the morphism $f : X \to \mathbb{P}^1$ is induced by the field inclusion $k(t) \to K(X)$ sending t to f, and $\operatorname{ord}_{(0)} t = 1$, so the definition of ramification index gives us precisely that the part of $D(f)$ with positive coefficients are the ramification indices of zeroes of f. But similarly, $\operatorname{ord}_\infty \frac{1}{t} = 1$, and maps to $\frac{1}{f}$, so the negative part of $D(f)$ is given by $\operatorname{ord}_P(\frac{1}{f}) = -\operatorname{ord}_P(f)$ at points P with $f(P) = \infty$.

The desired statement then follows immediately from Corollary 9.2.3, since $\deg([0] - [\infty]) = 0$. $\square$

Example 9.2.7. Consider the case that $X = \mathbb{P}^1$, and write $\mathbb{P}^1 = \mathbb{A}^1 \cup \{\infty\}$, with coordinate t on $\mathbb{A}^1$. Then a rational function $f \in K(X)^*$ is a quotient of polynomials $g, h \in k[t]$. At any point $\lambda \in \mathbb{A}^1$, the coefficient of $[\lambda]$ in $D(f)$ is simply the difference of the orders of vanishing of g and h at $t = \lambda$. On the other hand, at ∞ we have that $\frac{1}{t}$ is a local coordinate, and one then checks that the coefficient of $[\infty]$ in $D(f)$ is equal to $\deg h - \deg g$. The sum of the coefficients of points on $\mathbb{A}^1$ is $\deg g - \deg h$, so we see that the degree of $D(f)$ is 0, as asserted by Proposition 9.2.6.

We use divisors to study rational functions on a curve by considering all functions which vanish to certain prescribed orders at some points, and are allowed to have poles of certain orders at others. Formally, we have the following definition.

Definition 9.2.8. Given a divisor D on X, define

$$\mathcal{L}(D) = \{f \in K(X)^* : D(f) + D \geqslant 0\} \cup \{0\}.$$

Thus, if $D = \sum_i c_i[P_i] - \sum_j d_j[Q_j]$ where $c_i, d_j > 0$, and $P_i \neq Q_j$ for any i, j, then $\mathcal{L}(D)$ is the space of all rational functions which vanish to order at least d_j at each Q_j, but are allowed to have poles of order at most c_i at each P_i. This is visibly a k-vector space, and we will next prove that it is finite-dimensional when X is projective.

Lemma 9.2.9. *Given a divisor D on X, and $P \in X$ any point, the quotient space $\mathcal{L}(D)/\mathcal{L}(D - [P])$ has dimension at most* 1.

Proof. First observe that $\mathcal{L}(D - [P])$ is indeed a subspace of $\mathcal{L}(D)$, consisting precisely of those rational functions vanishing to order (possibly negative) strictly greater than required to be in $\mathcal{L}(D)$. The quotient vector space thus makes sense. If $\mathcal{L}(D - [P]) = \mathcal{L}(D)$ (which can certainly occur), there is nothing to prove. On the other hand, if $\mathcal{L}(D - [P]) \subsetneq \mathcal{L}(D)$, let f be in $\mathcal{L}(D) \smallsetminus \mathcal{L}(D - [P])$, and let t be a local coordinate on X at P. Then we know that $f = t^e g$ for some $e \in \mathbb{Z}$, and some g regular and nonvanishing in a neighborhood of P. Since $f \notin \mathcal{L}(D - [P])$, we see that $-e$ must be the coefficient of $[P]$ in D. We claim that f spans $\mathcal{L}(D)/\mathcal{L}(D - [P])$. Indeed, for any other $f' \in \mathcal{L}(D)$, we can write $f' = t^{e'} g'$ with g' regular and nonvanishing at P, and we must have $e' - e \geqslant 0$, so $e' \geqslant e$. Then we observe that

$$f' - \frac{t^{e'-e}(P) g'(P)}{g(P)} f$$

vanishes to order strictly greater than e at P, so is therefore in $\mathcal{L}(D - [P])$, proving our claim and the lemma. $\square$

Corollary 9.2.10. *For any divisor D on a projective nonsingular curve X, we have that $\mathcal{L}(D)$ is finite-dimensional over k, and in fact*

$$\dim_k \mathcal{L}(D) \leqslant \deg D + 1.$$

Proof. We see immediately from Proposition 9.2.6 that if $\deg D < 0$, then $\mathcal{L}(D) = 0$. The statement then follows immediately from Lemma 9.2.9 by induction on $\deg D$. $\square$

This dimension will be very important to us, so we give it its own notation.

Notation 9.2.11. We denote by $\ell(D)$ the dimension of $\mathcal{L}(D)$ over k.

Example 9.2.12. Again consider the case $X = \mathbb{P}^1$, continuing with the notation of Example 9.2.7, and let $D = d[\infty]$. Then if $f \in K(X)^*$ is written as $\frac{g}{h}$, where g, h have no common factors, we see that for f to be in $\mathcal{L}(D)$, we must have h constant, since we cannot have any poles away from ∞. Then we have $\deg h - \deg g = -\deg g \leqslant -d$, so we conclude that f must be a polynomial of degree at most d. Conversely, any polynomial of degree less than or equal to d is in $\mathcal{L}(D)$, so we see that $\ell(D) = d + 1$. In particular, sometimes the bound of Corollary 9.2.10 is achieved.

9.3. Linear equivalence and morphisms to projective space

Closely related to the study of divisors and rational functions is the study of morphisms to projective space. In this section, we suppose throughout that X is a nonsingular projective curve.

Suppose we have a morphism $\varphi : X \to \mathbb{P}^r$, which we assume to be **non-degenerate**, meaning that $\varphi(X)$ is not contained in any hyperplane H of $\mathbb{P}^r$. We then see that for any such H, we have $H \cap \varphi(X)$ a proper closed subset of $\varphi(X)$, hence a finite set of points. In fact, there is a natural way to associate an effective divisor on X to $H \cap \varphi(X)$. Suppose $P \in X$ such that $\varphi(P) \in H$, and choose i such that $\varphi(P) \in U_i = \mathbb{P}^r \smallsetminus Z(X_i)$. If $H = Z(\sum_j c_j X_j)$, then we identify U_i with $\mathbb{A}^r$ in the usual way by setting coordinates $y_j = \frac{X_j}{X_i}$ for $j \neq i$, and on U_i we have $Z(H) = Z(\sum_j c_j y_j)$, where $y_i = 1$. Then $\sum_j c_j y_j$ is a regular function on U_i, so $\varphi^{-1}(\sum_j c_j y_j)$ is regular at P, and we take its order at P to determine the coefficient of P in the divisor associated to $H \cap \varphi(X)$. One checks easily that this is independent of the choice of i and of the equation for H (which is unique up to scaling).

Notation 9.3.1. If $\varphi : X \to \mathbb{P}^r$ is a nondegenerate morphism, and $H \subseteq \mathbb{P}^r$ a hyperplane, we denote by $\varphi^*(H)$ the effective divisor on X associated to $H \cap \varphi(X)$.

Obviously, $\varphi^*(H)$ depends on H. However, if we choose a different hyperplane H', we find that $\varphi^*(H)$ and $\varphi^*(H')$ are closely related.

Proposition 9.3.2. *Given $\varphi : X \to \mathbb{P}^r$ a nondegenerate morphism, and $H, H' \subseteq \mathbb{P}^r$ two hyperplanes, then $\varphi^*(H) - \varphi^*(H') = D(f)$ for some rational function f on X.*

Proof. The basic idea is that if $H = Z(\sum_i c_i X_i)$, and $H' = Z(\sum_i c_i' X_i)$, then even though the defining equations do not give functions on $\mathbb{P}^r$, the quotient $\frac{\sum_i c_i X_i}{\sum_i c_i' X_i}$ defines a rational function on $\mathbb{P}^r$, which by non-degeneracy is regular and non-vanishing on a nonempty open subset of $\varphi(X)$, and thus pulls back under φ to give a rational function f on X. We need only verify that $D(f) = \varphi^*(H) - \varphi^*(H')$.

However, following the notation of the above discussion, if the ith coordinate of P is non-zero, we can divide through both the numerator and denominator by X_i and find

$$\frac{\sum_j c_j X_j}{\sum_j c_j' X_j} = \frac{\sum_j c_j y_j}{\sum_j c_j' y_j},$$

so the definition of $D(f)$ is visibly equal to $\varphi^*(H) - \varphi^*(H')$. $\square$

This motivates the following definition:

Definition 9.3.3. Two divisors D, D' on X are **linearly equivalent** if $D - D' = D(f)$ for some rational function f on X.

From Proposition 9.2.6, we see immediately:

Corollary 9.3.4. *Two linearly equivalent divisors have the same degree.*

We can thus define:

Definition 9.3.5. The **degree** of a nondegenerate morphism $\varphi : X \to \mathbb{P}^r$ is $\deg \varphi^*(H)$ for any hyperplane $H \subseteq \mathbb{P}^r$.

Remark 9.3.6. In fact, if $\varphi(X)$ is injective, then the degree is equal to the number of points in $\varphi(X) \cap H$ for a "sufficiently general" hyperplane H. That is, if H is not too special, all the points of $\varphi^*(H)$ will have multiplicity 1.

There is one situation of overlap between this definition and our earlier definition of degree for morphisms between curves. We verify that the two definitions agree in this case.

Proposition 9.3.7. *Let $\varphi : X \to \mathbb{P}^1$ be a nonconstant morphism. Then* $\deg \varphi^* H = [K(X) : K(\mathbb{P}^1)]$*, where H is any hyperplane (that is to say, point) in $\mathbb{P}^1$.*

Proof. This is not obvious from the definitions, but it follows easily from Theorem 9.1.5. Indeed, we see immediately from the definitions that

$$\varphi^* H = \sum_{P \in \varphi^{-1}(H)} e_P [P],$$

so the desired identity follows. $\square$

Linear equivalence also arises in the spaces $\mathcal{L}(D)$. If $f \in K(X)^*$ is in $\mathcal{L}(D)$, then instead of looking at the zeroes and poles of f as a rational function, we could ask what its "extra vanishing" is as an element of $\mathcal{L}(D)$; that is, we could look at the divisor $D(f) + D$, which is effective by definition. We have:

PROPOSITION 9.3.8. *Given a divisor D, the set $D(f) + D$ for nonzero $f \in \mathcal{L}(D)$ is precisely the set of effective divisors linearly equivalent to D.*

PROOF. Suppose D' is effective, and linearly equivalent to D. Then by definition, there is some $f \in K(X)^*$ with $D' - D = D(f)$, or equivalently, $D' = D(f) + D$. Thus $f \in \mathcal{L}(D)$, and we get one inclusion. Conversely, we have already observed that $D(f)+D$ is effective by definition if $f \in \mathcal{L}(D)$ is nonzero, but it is also visibly linearly equivalent to D. $\square$

We can thus think of the following definition in terms of families of effective, linearly equivalent divisors:

DEFINITION 9.3.9. A **linear series** is a vector subspace of $\mathcal{L}(D)$ for some D. A linear series is **complete** if it is equal to all of $\mathcal{L}(D)$. We say that the linear series has **degree** d and **rank** r if $\deg D = d$, and the subspace has dimension $r + 1$.

Associated to a linear series we get a family of effective divisors, but the linear series language is useful because the condition of being a subspace is easier to understand than the corresponding condition on the associated divisors. The discrepancy of 1 in the rank terminology will be explained shortly.

We now return to considering morphisms to projective space. As above, we suppose we have $\varphi : X \to \mathbb{P}^r$. We observe that the associated family of divisors $\varphi^*(H)$ for hyperplanes H in $\mathbb{P}^r$ have the property that there is no point $P \in X$ such that $[P]$ appears with positive coefficients in every $\varphi^*(H)$: indeed, we can choose H to be any hyperplane in $\mathbb{P}^r$ such that $\varphi(P) \notin H$, and then the coefficient of $[P]$ in $\varphi^*(H)$ will be 0.

DEFINITION 9.3.10. Given a linear series $V \subseteq \mathcal{L}(D)$, a point $P \in X$ is a **basepoint** of V if for all nonzero $f \in V$, the coefficient of $[P]$ in $D(f) + D$ is strictly positive. The linear series V is **basepoint-free** if no $P \in X$ is a base point of V.

LEMMA 9.3.11. *Suppose that D and D' are linearly equivalent divisors on a projective nonsingular curve X. Then there is an isomorphism $\alpha : \mathcal{L}(D) \to \mathcal{L}(D')$, unique up to scaling, such that $D(\alpha(f)) + D' = D(f) + D$ for all nonzero $f \in \mathcal{L}(D)$.*

PROOF. By definition, there is some $g \in K(X)^*$ with $D - D' = D(g)$. Because X is projective, only constant functions have $D(g) = 0$, we see that g is unique up to scaling by a (nonzero) constant. Multiplication by g then defines the desired isomorphism $\mathcal{L}(D) \to \mathcal{L}(D')$. $\square$

DEFINITION 9.3.12. We say that two linear series $V \subseteq \mathcal{L}(D)$ and $V' \subseteq \mathcal{L}(D')$ are **equivalent** if D is linearly equivalent to D' and V is mapped to V' under the isomorphism of Lemma 9.3.11.

PROPOSITION 9.3.13. *Linear series V and V' are equivalent if and only if the sets of effective divisors $\{D(f) + D : f \in V \smallsetminus \{0\}\}$ and $\{D(f) + D' : f \in V' \smallsetminus \{0\}\}$ are equal.*

PROOF. It is clear from the definition and Lemma 9.3.11 that if V and V' are equivalent, the sets of effective divisors are equal. Conversely, if the sets of effective divisors have any elements in common, we immediately conclude that D is linearly equivalent to D', and because the isomorphism α of Lemma 9.3.11 doesn't change the corresponding effective divisors $D(f) + D$, and $D(f) + D$ determines f up to nonzero scalar, we conclude that if the two sets of divisors are equal, then α must map V into V'. $\square$

The main theorem relating linear series to morphisms to projective space is the following:

THEOREM 9.3.14. *Let $V \subseteq \mathcal{L}(D)$ be a basepoint-free linear series on X of rank r and degree d. Then V is associated to a nondegenerate morphism $\varphi : X \to \mathbb{P}^r$, unique up to linear change of coordinate on $\mathbb{P}^r$, such that the set $D(f) + D$ for nonzero $f \in V$ is equal to the set of $\varphi^*(H)$ for hyperplanes $H \subseteq \mathbb{P}^r$. In particular, φ has degree d.*

Moreover, this construction gives a bijection between equivalence classes of linear series of rank r and degree d on X, and nondegenerate morphisms $\varphi : X \to \mathbb{P}^r$ of degree d, up to linear change of coordinates.

LEMMA 9.3.15. *Suppose a morphism $\varphi : X \to \mathbb{P}^r$ is described on an open subset $U \subseteq X$ by a tuple $(f_0, \dots, f_r)$ of functions regular and not simultaneously vanishing on U. Then φ is nondegenerate if and only if the f_i are linearly independent in $K(X)$ over k, and if $H \subseteq \mathbb{P}^r$ is the hyperplane $Z(\sum_i c_i X_i)$ for some $c_i \in k$, and*

$$D = -\sum_{P \in X} \min_i(\mathrm{ord}_P(f_i))[P],$$

then

$$\varphi^*(H) = D(\sum_i c_i f_i) + D.$$

PROOF. The assertion of nondegeneracy is clear, since if $\varphi(U)$ were contained in a hyperplane, we would obtain a linear dependence among the f_i, and conversely. To check the desired identity of divisors, fix $P \in X$, and suppose the ith coordinate of $\varphi(P)$ is nonzero; then the coefficient of $[P]$ in $\varphi^*(H)$ is defined to be the order of vanishing at P of $\varphi^*(\sum_j c_j \frac{X_j}{X_i})$. Let e be the coefficient of $[P]$ in D. Then each f_j has order at least $-e$ at P by hypothesis, and in fact at least one f_j has order exactly $-e$.

We know that φ is defined at P by multiplying all the f_j by t^e, for a local coordinate t at P, so that near P, the morphism φ is given by $(t^e f_0, \dots, t^e f_n)$, and so we see in particular that f_i has minimum order $-e$ at P. Furthermore, near P we have

$$\varphi^*(\frac{X_j}{X_i}) = \frac{t^e f_j}{t^e f_i} = \frac{f_j}{f_i},$$

so the coefficient of $[P]$ in $\varphi^*(H)$ is

$$\mathrm{ord}_P(\sum_j c_j \frac{f_j}{f_i}) = \mathrm{ord}_P(\sum_j c_j f_j) + e,$$

proving the desired statement. □

PROOF OF THEOREM 9.3.14. If we choose a basis $f_0, \dots, f_r$ for V, by linear independence and Lemma 9.3.15, the tuple $(f_0, \dots, f_r)$ defines a nondegenerate morphism $\varphi : X \to \mathbb{P}^r$, which satisfies the desired relationship. Now, in constructing φ, we chose the basis $\{f_i\}$ for V. It is clear that change of basis corresponds precisely to to modifying φ by composing $\mathbb{P}^r$ by the corresponding linear change of coordinates. We thus wish to prove that aside from this ambiguity, φ is uniquely determined by the set of $\varphi^*(H)$. But we see that if $\varphi : X \to \mathbb{P}^r$ is any nondegenerate morphism, we obtain effective divisors $D_0, \dots, D_n$ with $D_i := \varphi^*(Z(X_i))$, which by hypothesis are each of the form $D(g_i) + D$ for some $g_i \in \mathcal{L}(D)$. But if φ is given by $(f_0, \dots, f_r)$, and we let $D' = -\sum_{P \in X} \min_i(\mathrm{ord}_P(f_i))$, then by Lemma 9.3.15 we see that $D(f_i) + D' = D(g_i) + D$ for each i, so $D(f_i/g_i) = D - D'$. If we replace each g_i by $\frac{f_0}{g_0} g_i$, then we don't change φ, and we have $D(f_i) = D(g_i)$, so the f_i and g_i are each related by nonzero scalars, and we conclude the desired uniqueness.

Now, because the associated morphisms φ are characterized by the effective divisors $D(f)+D$, we immediately see that equivalent linear series yield the same (equivalence classes of) morphisms to projective space. For the final bijectivity assertion, it is thus enough to show that every nondegenerate morphism to projective space arises in the manner described. However, every morphism $\varphi : X \to \mathbb{P}^r$ is described on some open subset U by a tuple $(f_0, \dots, f_r)$ of regular functions on U, and from Lemma 9.3.15 we then see that if we set D as in the lemma statement, the f_i span a linear series in $\mathcal{L}(D)$ which satisfies the desired conditions. $\square$

EXERCISE 9.3.16. Let $X \subseteq \mathbb{P}^n_k$ be a nonsingular projective variety, and $P \in \mathbb{P}^n_k \smallsetminus X$. Then the linear projection from P defines a morphism $\varphi : X \to \mathbb{P}^{n-1}_k$.

(a) In terms of the geometry of X in $\mathbb{P}^n_k$, describe which points Q have the property that the induced map $T_Q(X) \to T_{\varphi(Q)}\mathbb{P}^{n-1}_k$ is injective.

(b) Assuming that φ is nondegenerate and X is a curve, describe the same set of points as in (a) in terms of the family of divisors on X corresponding to φ.

EXERCISE 9.3.17. Given $X \subseteq \mathbb{P}^2_k$ a nonsingular projective plane curve, and $P \in \mathbb{P}^2_k \smallsetminus X$, the linear projection from P defines a morphism $\varphi : X \to \mathbb{P}^1_k$.

(a) In terms of plane geometry, describe the ramification points of φ and their ramification indices.

(b) If $P = (0,1,0)$, describe the ramification points of φ in terms of a suitable partial derivative.

EXERCISE 9.3.18. Suppose that $F, G \in k[X_0, X_1, X_2]$ are homogeneous polynomials with no common factors.

(i) Given $P \in Z(F) \cap Z(G)$ suppose that $P \in U_i \subseteq \mathbb{P}^2_k$. Show that $\dim_k \mathscr{O}_{P,\mathbb{P}^2_k}/(d_i(F), d_i(G))$ is finite, and independent of the choice of i (Notation as in 6.1.3).

If F and G have no multiple factors, we let $X = Z(F)$ and $Y = Z(G)$ be the corresponding (possibly reducible) projective plane curves. In this case, we define the **intersection multiplicity** of X and Y at P to be

$$i_P(X \cdot Y) := \dim_k \mathscr{O}_{P,\mathbb{P}^2_k}/(d_i(F), d_i(G)).$$

(ii) Suppose further that X is nonsingular and F is irreducible. Given any homogeneous polynomial G' such that $X \not\subseteq Z(G')$, and $P \in X \cap Z(G')$, show that $\dim_k \mathscr{O}_{P,\mathbb{P}^2_k}/(d_i(F), d_i(G')) = \operatorname{ord}_P(d_i(G')|_X)$.

(iii) Under the hypotheses of (b), define a divisor $D(G'|_X)$ on X by

$$D(G'|_X) = \sum_{P \in X \cap Z(G')} \operatorname{ord}_P(d_i(G')|_X)[P].$$

Show that if G'' satisfies the same hypotheses as G', and has the same degree, then $D(G''|_X)$ is linearly equivalent to $D(G'|_X)$.

EXERCISE 9.3.19. Suppose $X \subseteq \mathbb{P}^2_k$ is a nonsingular plane curve, with $X = Z(F)$ for F homogeneous of degree d. Show that the inclusion $X \hookrightarrow \mathbb{P}^2_k$ corresponds to a linear series on X of degree d.

EXERCISE 9.3.20. (A special case of Bezout's theorem) Suppose that $F, G \in k[X_0, X_1, X_2]$ are distinct, irreducible and homogeneous, of degrees d, e respectively. Write $X = Z(F)$, $Y = Z(G)$, and suppose further that X is nonsingular. Then

$$\sum_{P \in X \cap Y} i_P(X \cdot Y) = d \cdot e.$$

The same formula holds even if X is not nonsingular; see for instance Corollary 7.8 of [**Har77**].

9.4. Imbeddings of curves

We consider when a morphism defines an isomorphism onto its image. We have:

THEOREM 9.4.1. *Let $\varphi : X \to Y$ be a morphism of prevarieties. Then φ induces an isomorphism onto a closed subset of Y if and only if the following conditions are satisfied:*

(i) *φ is injective;*
(ii) *φ induces injective maps $T_P(X) \to T_{\varphi(P)}(Y)$ of tangent spaces for all $P \in X$;*
(iii) *Y has an open cover $\{V_i\}$ by affine varieties such that for each i, the open subset $\varphi^{-1}(V_i)$ is affine, and the induced ring homomorphism makes $A(\varphi^{-1}(V_i))$ into a finitely generated $A(V_i)$-module.*

PROOF. Let $Z \subseteq Y$ be a closed subset containing $\varphi(X)$. Then for any affine open subset $U \subseteq Y$, we have that $Z \cap U$ is also affine, with $A(Z \cap U)$ being a quotient ring of $A(U)$.

Now, suppose that $Z = \varphi(X)$ and φ induces an isomorphism onto Z. Then we see immediately that (i)-(iii) are satisfied. Conversely, suppose (i)-(iii) are satisfied, and let $Z \subseteq Y$ be the closure of $\varphi(X)$. Then (i) and (ii) are also satisfied for the induced morphism $X \to Z$. Additionally, because $A(V_i \cap Z)$ is a quotient of $A(V_i)$, we see that generators of $A(\varphi^{-1}(V_i))$ over $A(V_i)$ will still be generators over $A(V_i \cap Z)$, so (iii) is also satisfied if we replace Y by Z. Thus, it suffices to prove the theorem in the case φ is dominant, in which case we simply wish to prove that φ is an isomorphism. It is then enough to show that for each i, the induced morphism $\varphi^{-1}(V_i) \to V_i$ is an isomorphism. We know that it is injective by (i). Moreover, we know it is surjective by (iii) and Proposition 7.3.6. It is then enough to show that the induced homomorphisms $\varphi_P : \mathscr{O}_{Y,\varphi(P)} \to \mathscr{O}_{X,P}$ are isomorphisms for all $P \in X$. Because φ is dominant, each φ_P is injective. Furthermore, because φ is injective, we have that

$$\mathscr{O}_{X,P} = A(\varphi^{-1}(U_i))_{\mathfrak{m}_P} = A(\varphi^{-1}(U_i))_{\mathfrak{m}_{\varphi(P)}},$$

where the last term means that we consider $A(\varphi^{-1}(U_i))$ as an $A(U_i)$-module, and invert the complement of $\mathfrak{m}_{\varphi(P)}$. Since $\mathscr{O}_{Y,\varphi(P)} = A(U_i)_{\mathfrak{m}_{\varphi(P)}}$, this means that generators of $A(\varphi^{-1}(U_i))$ over $A(U_i)$ also induce generators of $\mathscr{O}_{X,P}$ over $\mathscr{O}_{Y,\varphi(P)}$. The desired statement then follows from the below algebra lemma together with hypotheses (ii) and (iii). $\square$

LEMMA 9.4.2. *Let $f : R \to S$ be a homomorphism of Noetherian local rings, such that*

(i) *$f(\mathfrak{m}_R) \subseteq \mathfrak{m}_S$;*
(ii) *the map $R/\mathfrak{m}_R \to S/\mathfrak{m}_S$ induced by f is an isomorphism;*
(iii) *the map $\mathfrak{m}_R/\mathfrak{m}_R^2 \to \mathfrak{m}_S/\mathfrak{m}_S^2$ induced by f is surjective;*
(iv) *f makes S into a finitely-generated R-module.*

Then f is surjective.

This is a relatively easy application of Nakayama's lemma; see Lemma 7.4 of Chapter II of [**Har77**].

Applying Theorem 9.4.1 together with Corollary 7.4.6, we conclude the following.

COROLLARY 9.4.3. *Let X be a projective nonsingular curve, Y a variety, and $\varphi : X \to Y$ a morphism. Then φ induces an isomorphism of X onto a closed subset of Y if and only if φ is injective, and induces injective maps on tangent spaces at all points of X.*

REMARK 9.4.4. We see that projectivity is a crucial hypothesis in Corollary 9.4.3. Otherwise, let Y be a curve with one node, let $\widetilde{Y} \to Y$ be the normalization, and let $X \subseteq \widetilde{Y}$ be the open subset omitting one of the two points of $\widetilde{Y}$ lying over the node of Y. Then it is straightforward to check that the morphism $X \to Y$ is bijective and induces injections on tangent spaces, but φ is not an isomorphism.

In contrast, the hypothesis that X is a nonsingular curve is not necessary (see Proposition 7.3 of Chapter II of [**Har77**] for a closely related statement), and we make use of it only to simplify the proof that hypothesis (iii) of Theorem 9.4.1 is satisfied. In fact, the latter hypothesis is satisfied for any morphism $\varphi : X \to Y$ where X is projective, Y is a variety, and φ has finite fibers. However, the proof is quite a bit more involved; compare Theorem 5.19 of Chapter II of [**Har77**]. Furthermore, instead of taking X projective, it suffices to suppose X is complete, but this makes the proof even more difficult.

We now rephrase Corollary 9.4.3 in the special case of morphisms to projective space.

COROLLARY 9.4.5. *Let X be a projective nonsingular curve, and $V \subseteq \mathcal{L}(D)$ a basepoint-free linear series of rank r on X. Then the induced morphism $\varphi : X \to \mathbb{P}^r_k$ gives an isomorphism of X onto a closed curve in $\mathbb{P}^r_k$ if and only if for all $P, Q \in X$ (possibly equal), we have*

$$\dim_k(V \cap \mathcal{L}(D - [P] - [Q])) = \dim_k V - 2.$$

By Lemma 9.2.9, we see that for any P, Q, we have $\dim_k(V \cap \mathcal{L}(D-[P]-[Q])) \geqslant \dim_k V - 2$, so the hypothesis is that $V \cap \mathcal{L}(D-[P]-[Q])$ is as small as possible for all P, Q. Additionally, from the definition of a base point (and Lemma 9.2.9 again), we have that $\dim_k(V \cap \mathcal{L}(D-[P])) = \dim_k V - 1$ for any $P \in X$, so the condition can be rephrased as saying that $V \cap \mathcal{L}(D-[P]-[Q]) \neq V \cap \mathcal{L}(D-[P])$ for all $P, Q \in X$. When P, Q are distinct, this is equivalent to the condition that $\varphi(P) \neq \varphi(Q)$, while when $P = Q$, this turns out to be the same as injectivity on tangent spaces.

9.5. Secant varieties and curves in projective space

We have seen that every nonsingular curve is quasiprojective. We now elaborate on this by studying what kind of morphisms we can construct into specific projective spaces. Specifically, we will prove:

PROPOSITION 9.5.1. *Suppose C is a nonsingular curve. Then there exists an injective morphism $\varphi : C \to \mathbb{P}^3$. Given any $Q \in C$, there also exists a morphism $\varphi' : C \to \mathbb{P}^2$ such that*

$$(\varphi')^{-1}(\varphi'(Q)) = \{Q\}.$$

This statement will be useful in analyzing the topology of complex varieties, and it follows immediately from the following two more general results.

PROPOSITION 9.5.2. *Suppose X is a quasiprojective variety of dimension d. Then there exists an injective morphism $\varphi : X \to \mathbb{P}^{2d+1}$.*

PROPOSITION 9.5.3. *Given $Q \in X$ any point on a quasiprojective variety of dimension d, there exists a morphism $\varphi' : X \to \mathbb{P}^{d+1}$ such that $(\varphi')^{-1}(\varphi'(Q)) = \{Q\}$.*

The idea of the argument is that we start with a realization of X as a subvariety of some $\mathbb{P}^n$, and then show we can project inductively to smaller-dimensional projective spaces. The key construction is:

DEFINITION 9.5.4. Given a subvariety $X \subseteq \mathbb{P}^n$, the **secant variety** $\mathrm{Sec}(X)$ of X is the Zariski closure of the set of points P of $\mathbb{P}^n$ such that there exist distinct points $Q_1, Q_2 \in X$ with P lying on the line through Q_1, Q_2.

PROPOSITION 9.5.5. *The secant variety of X is a variety, of dimension less than or equal to $2 \dim X + 1$.*

Proof. Let d be the dimension of X. We first consider the auxiliary variety $\widetilde{\mathrm{Sec}}(X) \subseteq \mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^n$ defined by triples of points (Q_1, Q_2, P) with $Q_1, Q_2 \in X$ distinct, and P on the line between Q_1 and Q_2. By definition, $\mathrm{Sec}(X)$ is the closure of the image of $\widetilde{\mathrm{Sec}}(X)$ under the third projection morphism. It thus suffices to prove that $\widetilde{\mathrm{Sec}}(X)$ is a variety of dimension equal to $2d+1$.

We first see that it is an open subset of a closed subset of $\mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^n$. Indeed, it is the intersection of $X \times X \times \mathbb{P}^n$ with the set of triples of points $(Q_1, Q_2, P) \in (\mathbb{P}^n \times \mathbb{P}^n \smallsetminus \Delta(\mathbb{P}^n)) \times \mathbb{P}^n)$ such that P lies on the line between Q_1 and Q_2. The latter set can be described by the condition that the coordinate vectors of Q_1, Q_2 and P are linearly dependent, which is a polynomial condition in the coordinates of the three points given by the vanishing of the 3×3 minors of the associated $3 \times (n+1)$ matrix (the matrix is defined only up to nonzero scaling of each row, but this will not affect whether or not a given minor vanishes). Thus $\widetilde{\mathrm{Sec}}(X)$ is an open subset of a closed subset, and it suffices to see that it is irreducible of dimension $2d+1$.

We prove both statements at once as follows: let $U_0, \dots, U_{n+1}$ be the affine open cover of X with $U_i = X \smallsetminus Z(x_i)$. For any i, j, let $U_{i,j} \subseteq \widetilde{\mathrm{Sec}}(X)$ be the preimage of $U_i \times U_j$ under the projection morphism to the first two factors; that is,

$$U_{i,j} = \{(Q_1, Q_2, P) \in \widetilde{\mathrm{Sec}}(X) : Q_1 \in U_i, Q_2 \in U_j\}.$$

It is clear that the $U_{i,j}$ form an open cover of $\widetilde{\mathrm{Sec}}(X)$. We claim that $U_{i,j} \cong (U_i \times U_j \smallsetminus \Delta(X)) \times \mathbb{P}^1$. For each U_i, we normalize coordinates in $\mathbb{P}^n$ so that $x_i = 1$. We then obtain a morphism

$$\psi_{i,j} : (U_i \times U_j \smallsetminus \Delta(X)) \times \mathbb{P}^1 \to U_{i,j}$$

by sending $((x_0, \dots, x_n), (y_0, \dots, y_n), (s,t))$ to $(x_0, \dots, x_n), (y_0, \dots, y_n), (sx_0 + ty_0, \dots, sx_n + ty_n)$. Note that $\psi_{i,j}$ is well defined because we have normalized coordinates on U_i, U_j and simultaneously scaling (s,t) will scale the last $(n+1)$-tuple. Furthermore, $\psi_{i,j}$ is visibly bijective, so we wish to show that the inverse is a morphism. In order to do so, we may restrict to the open cover $V_{a,b}$ of $U_{i,j}$ consisting of points $((x_0, \dots, x_n), (y_0, \dots, y_n), (z_0, \dots, z_n))$ on which (x_a, x_b) is linearly independent from (y_a, y_b). Recall that $(x_0, \dots, x_n) \neq (y_0, \dots, y_n)$, so this does in fact form a cover. On $V_{a,b}$, we see that $\psi_{i,j}^{-1}$ is expressed by setting

$$s = y_b z_a - y_a z_b, t = x_a z_b - x_b z_a;$$

after composition, (s,t) is scaled by $x_a y_b - x_b y_a$, which doesn't change the point in $\mathbb{P}^1$. Thus $\psi_{i,j}^{-1}$ is also a morphism, and $\psi_{i,j}$ is an isomorphism.

Since $U_i \times U_j \times \mathbb{P}^1$ is a variety of dimension $2d+1$, we conclude the same for $U_{i,j}$. Since $\widetilde{\mathrm{Sec}}(X)$ is covered by the $U_{i,j}$, the only point that remains is to check irreducibility. We observe that given i', j', we have $U_i \cap U_{i'} \neq \emptyset$ and $U_j \cap U_{j'} \neq \emptyset$, so $U_{i,j} \cap U_{i',j'} \neq \emptyset$. Since the $U_{i,j}$ form an open cover, and each is irreducible, we conclude that $\widetilde{\mathrm{Sec}}(X)$ does not contain any pair of disjoint non-empty open subsets, and thus is irreducible, as desired. $\square$

Remark 9.5.6. The technique used in the proof of Proposition 9.5.5 arises frequently in algebraic geometry. When one has a subset defined in terms of the existence of certain objects (as the secant variety is defined in terms of existence of the points Q_1, Q_2), it can be difficult to analyze the subset directly. Instead, we construct an auxiliary set in terms of choices of the objects in question, so that our original subset is the image of the new set under a natural forgetful map. Typically, the auxiliary set will be easier to analyze, and often better behaved (for instance, less singular).

Remark 9.5.7. Sometimes, $\mathrm{Sec}(X)$ is defined without taking the Zariski closure, but then it requires more care to check that it is in fact a subvariety.

We can now prove the main propositions.

Proof of Proposition 9.5.2. By hypothesis, we can realize $X \subseteq \mathbb{P}^n$ for some n. We prove by downward induction that there is an injective morphism $\varphi : X \hookrightarrow \mathbb{P}^{2d+1}$. It suffices to show that given an injective morphism $\varphi_m : X \to \mathbb{P}^m$, if $m > 2d + 1$ we can produce an injective morphism $\varphi_{m-1} : X \to \mathbb{P}^{m-1}$. Let X_m be the Zariski closure of $\varphi_m(X)$; then it is also a variety of dimension at most (in fact, exactly) d. Now choose $P \in \mathbb{P}^m \smallsetminus \operatorname{Sec}(X_m)$, which is nonempty since $m > 2d + 1$ and $\dim \operatorname{Sec}(X_m) \leqslant 2d + 1$ by Proposition 9.5.5. Let H be any hyperplane not containing P, and let $\varphi_{m-1} = \pi_P \circ \varphi_m$, where $\pi_P : \mathbb{P}^m \smallsetminus \{P\} \to H$ is the projection morphism sending a point Q to the point of H lying on the line through P and Q. Note that $X_m \subseteq \operatorname{Sec}(X_m)$, so π_P is defined on X_m. Since φ_m is injective by hypothesis, it suffices to see that $\pi_P|_{X_m}$ is injective. But two points $Q_1, Q_2 \in X_m$ have $\pi_P(Q_1) = \pi_P(Q_2)$ if and only if the line through P and Q_1 is the same as the line through P and Q_2, which can only happen if $Q_1 = Q_2$ by the hypothesis that $P \notin \operatorname{Sec}(X_m)$. Thus we have produced the desired φ_{m-1}. □

Proof of Proposition 9.5.3. We mimic the argument of Proposition 9.5.2. Suppose that $\varphi_m : X \to \mathbb{P}^m$ is a morphism such that $(\varphi_m)^{-1}(\varphi_m(Q)) = \{Q\}$, and $m > d + 1$ (as before, we can start with any imbedding of X in some $\mathbb{P}^n$). Arguing as in Proposition 9.5.5, it is clear that for any $Y \subseteq \mathbb{P}^m$ of dimension d, and $Q_1 \in Y$, the set of points $P \in \mathbb{P}^m$ such that P lies on the line connecting Q_1 to some $Q_2 \neq Q_1$ on X has dimension at most $d + 1$. Thus if we choose P not in this set, and H any hyperplane not containing P, and we set $\varphi_{m-1} = \pi_P \circ \varphi_m$, we find that φ_{m-1} satisfies $(\varphi_{m-1})^{-1}(\varphi_{m-1}(Q)) = \{Q\}$. Inducting downwards, we obtain the desired φ'. □

Remark 9.5.8. The nonsingularity hypothesis in Proposition 9.5.1 is in fact unnecessary; we include it only because we have not proved that arbitrary curves are quasiprojective. However, an elaboration of the techniques used above allows one to prove that every nonsingular curve is isomorphic to a curve in $\mathbb{P}^3$, and birational to a nodal curve in $\mathbb{P}^2$. These statements do require nonsingularity. See Chapter IV, Corollary 3.6 and Theorem 3.10 of [**Har77**].

CHAPTER 10

Differential forms

Differentials are an important topic in algebraic geometry, allowing the use of some classical geometric arguments in the context of varieties over any field. We will use them to define the genus of a curve, and to analyze the ramification of morphisms between curves. Although differentials remain important for arbitrary varieties, we will restrict our treatment to the case of nonsingular varieties. This treatment is taken primarily from Shafarevich [**Sha94a**].

10.1. Differential forms

A differential form is not a function, but can be defined in an analogous manner.

Definition 10.1.1. Let X be a nonsingular variety, and $U \subseteq X$ an open subset. A **differential form** on U associates to each point $P \in U$ an element of the Zariski cotangent space $T_P^*(X)$.

Differential forms as defined above play a role analogous to that of arbitrary functions: we need to restrict to a much smaller collection of them in order to obtain a useful concept. We do this by observing that for every regular function, we have an associated differential form.

Definition 10.1.2. Given $U \subseteq X$ an open subset of a nonsingular variety, and $f \in \mathscr{O}(U)$, the differential form df associated to f is defined as follows: for $P \in U$, let $df(P) \in \mathfrak{m}_P/\mathfrak{m}_P^2$ be the equivalence class of $(U, f - f(P))$.

A differential form ω on U is **regular** if for every $P \in U$, there exist an open neighborhood $V \subseteq U$ of P and regular functions $f_1, \dots, f_m, g_1, \dots, g_m \in \mathscr{O}(V)$ such that $\omega|_V = \sum_i f_i dg_i$.

Notation 10.1.3. We denote by $\Omega(U)$ the set of regular differential forms on U.

It is clear that $\Omega(U)$ is a module over $\mathscr{O}(U)$. Moreover, d defines a map $\mathscr{O}(U) \to \Omega(U)$ which is visibly k-linear, but not $\mathscr{O}(U)$-linear. Instead, we have the **Leibniz rule**:

Exercise 10.1.4. Show that for any $f, g \in \mathscr{O}(U)$, we have $d(fg) = fdg + gdf$.

This in turn gives us a chain rule for differential forms.

Exercise 10.1.5. Suppose that $g \in k(t_1, \dots, t_n)$, and $f_1, \dots, f_n$ are regular on $U \subseteq X$. Then away from the zero set of the denominator of g, we have

$$d(g(f_1, \dots, f_n)) = \sum_{i=1}^{n} \frac{\partial g}{\partial t_i}(f_1, \dots, f_n) df_i.$$

Because of the nonsingularity hypothesis, locally on X the modules of differential forms are free of rank equal to the dimension of X.

Lemma 10.1.6. *Given $P \in X$, if $(f_1, \dots, f_n)$ are a system of local coordinates for X at P, there exists an open set $U \ni P$ on which all the f_i are regular, and such that for every open subset $V \subseteq U$, every $\omega \in \Omega(V)$ can be written uniquely as*

$$\sum_i g_i df_i$$

for some $g_i \in \mathscr{O}(V)$. For every $Q \in U$, we have that the $f_i - f_i(Q)$ give a basis of $\mathfrak{m}_Q/\mathfrak{m}_Q^2$.

Proof. First let U' be an affine open neighborhood of P in X on which all the f_i are regular. Then extending $f_1, \dots, f_n$ to a set of generators of $A(U')$, we obtain an imbedding $U' \subseteq \mathbb{A}^m$ with coordinates $t_1, \dots, t_m$ such that $t_i|_{U'} = f_i$ for $i = 1, \dots, n$. Let $g_1, \dots, g_d$ be a set of generators of $I(U') \subseteq \mathbb{A}^m$. Then for each i, if we restrict to U' we have

$$0 = dg_i = \sum_j \frac{\partial g_i}{\partial t_j} dt_j.$$

Because X is nonsingular at P, we have by Corollary 4.3.2 that the rank of the Jacobian matrix $(\partial g_i/\partial t_j(P))$ is equal to $m - n$, and by our hypothesis that the $f_j = t_j|_{U'}$ generate $\mathfrak{m}_P/\mathfrak{m}_P^2$, we find that every $dt_j|_{U'}$ can be expressed in terms of $df_1, \dots, df_n$, with coefficients that are rational functions on X, regular at P. If we let $U \subseteq U'$ be an open neighborhood of P on which all the coefficient functions are regular, we claim that for any $V \subseteq U$ open, and $\omega \in \Omega(V)$, there exist unique $g_i \in \mathscr{O}(V)$ with

$$\omega = \sum_i g_i df_i.$$

We observe that at any point $Q \in U'$, the $dt_j|_{U'}$ for $j = 1, \dots, m$ span $\mathfrak{m}_Q/\mathfrak{m}_Q^2$, and it follows that if $Q \in U'$, in fact the df_j for $j = 1, \dots, n$ span $\mathfrak{m}_Q/\mathfrak{m}_Q^2$, so they must be a basis. We conclude that the desired g_i are unique, if they exist. On the other hand, since every regular function on any open subset of U' is a rational function in the t_i, using Exercise 10.1.5 we know that ω can be written locally near any point $Q \in V$ as a sum of the form

$$\sum_i h_i dt_i,$$

where the h_i are rational functions in the t_j, regular at Q. But we can similarly express each dt_i for $i > n$ as a combination of the $dt_1, \dots, dt_n$ with coefficients being rational functions in the t_j, regular at Q, so we obtain the desired express. □

It is then clear that we have:

Corollary 10.1.7. *Given f_i and U as in Lemma 10.1.6, we have that $\omega = \sum_i g_i df_i$ vanishes at $Q \in U$ if and only all the g_i vanish at Q.*

In particular, for any U open in X, the locus on which any regular differential form $\omega \in \Omega(U)$ vanishes is closed in U.

We conclude immediately from the second statement that regular differential forms satisfy the same rigidity property as regular functions.

Lemma 10.1.8. *Suppose $U \subseteq V \subseteq X$ are open subsets. If two regular differential forms on V are equal after restriction to U, then they are equal on V.*

We can thus define a rational differential form just as we did a rational function.

Definition 10.1.9. A **rational differential form** on X is an equivalence class of pairs (U, ω) where $U \subseteq X$ is open, and ω is a regular differential form on U. The equivalence relation is that $(U, \omega\rangle \sim (V, \omega')$ if $\omega|_{U \cap V} = \omega'|_{U \cap V}$.

The rational differential forms are clearly a vector space over $K(X)$. We conclude our general discussion of differentials with a description of the rational differential forms on X.

Proposition 10.1.10. *The rational differential forms on X have dimension over $K(X)$ equal to $\dim X$, and indeed if $P \in X$ is any point, and $(t_1, \dots, t_n)$ is a system of local coordinates for X at P, then $dt_1, \dots, dt_n$ form a basis of the rational differential forms on X over $K(X)$.*

Proof. We know from Lemma 10.1.6 that there exists an open neighborhood U of P on which every $\omega \in \Omega(U)$ can be written uniquely as $\sum_i f_i dt_i$ for $f_i \in \mathscr{O}(U)$, and that in fact the same holds for every $V \subseteq U$. But then the desired statement is clear, since every rational differential form has a representative on some $V \subseteq U$, as does every rational function. $\square$

10.2. Differential forms on curves

Just as with rational functions, if X is a nonsingular curve, and ω a rational differential form on X, we have a notion of order of zeroes or poles of ω at points on X, and we can associate a divisor $D(\omega)$ to ω. We assume throughout this section that X is a nonsingular.

Definition 10.2.1. If ω is a nonzero rational differential form on X, we define the associated divisor $D(\omega)$ on X as follows: for any $P \in X$, let t be a local coordinate, and write $\omega = f dt$ for some $f \in K(X)^*$. Then the coefficient of $[P]$ in $D(\omega)$ is $\operatorname{ord}_P(f)$.

Proposition 10.2.2. *The divisor $D(\omega)$ is a well-defined divisor on X. It has nonnegative coefficient at P if and only if ω is regular in a neighborhood of P, and strictly positive coefficient at P if and only if ω vanishes at P.*

Proof. By Proposition 10.1.10 we have that $\omega = f dt$ for a uniquely determined f, and by Lemma 10.1.8 if t, t' are two local coordinates, then dt and dt' can each be written as regular multiples of one another, so we must have $dt' = g dt$ for some g regular and nonvanishing at P, and we find that $D(\omega)$ is well-defined at each point.

If $D(\omega)$ has nonnegative coefficient at P, then f is regular at (and therefore in a neighborhood of) P, so ω is as well. Conversely, if ω is regular at P, we know from Lemma 10.1.6 that ω can be written as $f dt$ for f regular on a neighborhood of P, so $D(\omega)$ has nonnegative coefficient at P. Corollary 10.1.7 says that ω vanishes at P if and only if f does, and we know that $\operatorname{ord}_P(f) > 0$ if and only $f(P) = 0$.

Finally, we see that $D(\omega)$ is indeed a divisor, because it can have nonzero coefficient at only finitely many points: the points at which ω is not regular, and the points at which ω vanishes. $\square$

We can thus define particular spaces of rational differential forms subject to vanishing conditions:

Definition 10.2.3. Denote by $\Omega(D)$ the k-vector space of rational differential forms ω on X such that $\omega = 0$ or $D(\omega) + D \geqslant 0$.

We have the following easy consequence of the definition of $D(\omega)$:

Proposition 10.2.4. *Given any $f \in K(X)^*$ and nonzero rational differential form ω, we have*

$$D(f\omega) = D(f) + D(\omega).$$

Two key facts, both following from Proposition 10.1.10, are the following:

Corollary 10.2.5. *Given any two nonzero rational differential forms ω, ω' on X, we have that $D(\omega)$ and $D(\omega')$ are linearly equivalent.*

Proof. Given Proposition 10.2.4, this follows immediately from Proposition 10.1.10 which implies that $\omega' = f\omega$ for some $f \in K(X)^*$. $\square$

Corollary 10.2.6. *For any divisor D, the space $\Omega(D)$ is finite-dimensional over k.*

Proof. Let $K = D(\omega)$ for some nonzero rational differential form ω on X. By Proposition 10.1.10, every other such form ω' can be written uniquely as $f\omega$ for some $f \in K(X)^*$, and by Proposition 10.2.4 we have $D(\omega') = D(f) + K$. It follows immediately that $\Omega(D)$ is isomorphic to $\mathcal{L}(K + D)$ via the map $\omega' \mapsto f$. $\square$

In particular, we are now able to make the following fundamental definition.

Definition 10.2.7. If X is a nonsingular projective curve, we define the **genus** of X to be the dimension over k of $\Omega(X)$.

Example 10.2.8. If $X = \mathbb{P}^1$, let t be a coordinate on $\mathbb{A}^1 \subseteq \mathbb{P}^1$. Then a differential form regular on $\mathbb{A}^1$ is of the form fdt for some $f \in k[t]$, but one checks easily that dt has a pole of order 2 at ∞, and therefore no matter what f is, the form fdt cannot be regular at ∞. Thus $\mathbb{P}^1$ has genus 0. In fact, we will see later that up to isomorphism, $\mathbb{P}^1$ is the only nonsingular projective curve of genus 0.

10.3. Differential forms and ramification

We next want to study the structure of ramification of morphisms of curves.

Definition 10.3.1. A nonconstant morphism $\varphi : X \to Y$ of curves is **separable** if the induced field extension $K(X)/K(Y)$ is separable. Otherwise, we say φ is **inseparable**.

In particular, if k has characteristic 0, every (nonconstant) morphism is separable. We aim to prove a fundamental theorem relating ramification to separability. In order to so, we have to investigate the behavior of differential forms under morphisms. We make the following definition:

Definition 10.3.2. Given a morphism $\varphi : X \to Y$, and a differential form ω on $V \subseteq Y$, let $\varphi^*(\omega)$ be the **pullback** differential form on $\varphi^{-1}(V)$ induced by ω and φ using the linear maps $T^*_{\varphi(P)} \to T^*_P$ for every $P \in \varphi^{-1}(V)$.

Now, we see that

$$\varphi^* \sum_i g_i df_i = \sum_i \varphi^* g_i d\varphi^* f_i,$$

so if ω is regular, then $\varphi^*\omega$ is likewise regular. In particular, if φ is nonconstant, then φ^* gives a map from rational differential forms on Y to rational differential forms on X.

Theorem 10.3.3. *Let $\varphi : X \to Y$ be a nonconstant morphism of nonsingular curves. Then the following are equivalent:*

(a) *φ is inseparable;*
(b) *infinitely many points of X are ramifications points of φ;*
(c) *every $P \in X$ is a ramification point of φ;*
(d) *for every rational differential form ω on Y, we have $\varphi^*\omega = 0$.*

As suggested by the theorem statement, we will be interested in the behavior of pullback of differential forms for morphisms of nonsingular curves. An important preliminary definition is:

Definition 10.3.4. A nonconstant morphism $\varphi : X \to Y$ of curves is **wildly ramified** at P if $\operatorname{char} k = p > 0$ and $p | e_P$. If P is a ramification point at which φ is not wildly ramified, we say φ is **tamely ramified** at P. We say φ is **tamely ramified** if every P is either unramified or tamely ramified.

We can then state the relationship between ramification and pullback of differential forms as follows.

Proposition 10.3.5. *Given a nonconstant morphism $\varphi : X \to Y$ of nonsingular curves, and $P \in X$, and t a local coordinate at $\varphi(P)$, then P is a ramification point of φ if and only if $\varphi^* dt$ vanishes at P.*

More precisely, if $\varphi^* dt$ is not uniformly zero, then it vanishes to order at least $e_P - 1$ at P, and φ is wildly ramified at P if and only if we either have strict inequality, or $\varphi^* dt = 0$. In addition, the order of vanishing of $\varphi^* dt$ is independent of the choice of t.

Proof. By definition, we have $\varphi^* dt = d\varphi^* t$. On the other hand, $\varphi^* t = gs^{e_P}$, where s is a local coordinate at P, and g is a nonvanishing regular function on a neighborhood of P. Thus,

$$\varphi^* dt = d(gs^{e_P}) = s^{e_P} dg + e_P s^{e_P - 1} g ds.$$

We see that if this is nonzero, it vanishes to order at least $e_P - 1$, as claimed. Morever, $s^{e_P} dg$ vanishes to order at least e_P, and $s^{e_P-1} g ds$ vanishes to order exactly $e_P - 1$, so we conclude that $\varphi^* dt$ is either identically zero or vanishes to order strictly greater than $e_P - 1$ if and only if $e_P = 0$ in k, which is exactly the case of wild ramification. ☐

Remark 10.3.6. The first part of Proposition 10.3.5 can be rephrased as saying that φ is ramified at P if and only if the induced linear map $T_P(X) \to T_{\varphi(P)}(Y)$ is equal to 0.

The following notation isn't standard, but it will be convenient.

Notation 10.3.7. In the situation of Proposition 10.3.5, if $\varphi^* dt$ is not uniformly zero, write $\operatorname{dord}_P \varphi := \operatorname{ord}_P \varphi^* dt$.

The basic behavior of inseparable extensions in the case of transcendence degree 1 is the following:

Exercise 10.3.8. Suppose L/K is an algebraic extension of fields of characteristic $p > 0$, and $f \in L$ has a minimal polynomial $h(t) \in K[t]$ such that each coefficient of h is a pth power in K, then $f = g^p$ for some $g \in L$.

Proposition 10.3.9. *Given $f \in K(X) \smallsetminus k$, we have $df = 0$ if and only if* $\operatorname{char} k = p > 0$ *and $f = g^p$ for some $g \in K(X)$.*

Proof. If $f = g^p$, we have $df = pg^{p-1} dg = 0$ in characteristic p, by Exercise 10.1.5. For the converse, let t be a local coordinate at any point of X, so that we know from Proposition 10.1.10 that dt is a basis over $K(X)$ for the rational differential forms on X; in particular, $g dt = 0$ on any open subset if and only if $g = 0$. Now, $k(t)$ has transcendence degree 1, so $K(X)$ is algebraic over $k(t)$; in particular, f satisfies a polynomial relation $h(f) = 0$ for some $h \in k(t)[z]$. We may assume that h is the minimal polynomial of f, and in particular irreducible. Since $f \notin k$, and k is algebraically closed, we have that at least one coefficient of h is not in k. Clearing denominators if necessary, we may assume that the coefficients of h are in $k[t]$, with no common factors. Writing $h(z) = \sum_i h_i z^i$ and applying Exercise 10.1.5 (considering h as a polynomial in t and z) and the hypothesis that $df = 0$, we have

$$0 = d(h(f)) = \frac{dh}{dz}(f) df + \frac{dh}{dt}(f) dt = \sum_i \frac{dh_i}{dt} f^i dt,$$

so $\sum_i \frac{dh_i}{dt} f^i = 0$. We conclude that $\sum_i \frac{dh_i}{dt} z^i$ is a polynomial having f as a root, but it has at most the same degree as the minimal polynomial for f, and the degree in t of the coefficients is strictly smaller, which by uniqueness of the minimal polynomial is not possible unless $\frac{dh_i}{dt} = 0$ for all i. We conclude that each h_i has nonzero coefficients only for powers of t which are multiples of p; since k is algebraically closed, each h_i is a pth power. Thus, we conclude $f = g^p$ for some $g \in K(X)$ by Exercise 10.3.8. ☐

Exercise 10.3.10. Suppose k has characteristic $p > 0$, and K is finitely generated of transcendence degree 1 over k.

(a) Prove that K^p is a subfield of K, and K has degree p over K^p.

(b) Prove that if L is a subfield of K, also of transcendence degree 1 over k, then K/L is inseparable if and only if $L \subseteq K^p$.

Remark 10.3.11. The geometric content of Exercise 10.3.10 is that a nonconstant morphism $X \to Y$ of nonsingular curves is inseparable if and only if it factors through a certain Frobenius map $X \to X^{(p)}$.

We can now prove the theorem.

Proof of Theorem 10.3.3. It follows from Exercise 10.3.10 that if φ is inseparable, then for any $f \in K(Y)$, there exists $g \in K(X)$ such that $\varphi^* f = g^p$. By Proposition 10.3.9, we conclude that $\varphi^* df = d\varphi^* f = 0$; since f was arbitrary, it follows that $\varphi^* \omega = 0$ for all rational differential forms on Y, so (a) implies (d). Now, for any $P \in X$, applying this to the case that $\omega = dt$ for t a local coordinate at $\varphi(P)$, we see from Proposition 10.3.5 that P is a ramification point of X, so (d) implies (c).

(c) implies (b) trivially, so it remains to see that if φ is separable, it is ramified at only finitely many points of X. Let t be a local coordinate at some point $Q \in Y$; then since t has valuation 1 at Q, we have that t is not a pth power in $K(Y)$. It follows from separability of $K(X)$ over $K(Y)$ that $\varphi^* t$ is not a pth power in $K(X)$, and thus $\varphi^* dt = d\varphi^* t \neq 0$ by Proposition 10.3.9. Now, we know that $t - t(Q')$ is a local coordinate at Q' for Q' in some open neighborhood V of Q, so by Proposition 10.3.5, on $\varphi^{-1}(V)$ the ramification of φ is determined by the vanishing of $\varphi^* d(t - t(Q')) = \varphi^* dt$. But because $\varphi^* dt \neq 0$, it vanishes at only finitely many points of $\varphi^{-1}(V)$, and we conclude that φ can only be ramified at those points or in $X \smallsetminus \varphi^{-1}(V)$, which is also a finite set. $\square$

Since we cannot have inseparable extensions in characteristic 0, we conclude the following immediately from Theorem 10.3.3.

Corollary 10.3.12. *If* $\operatorname{char} k = 0$*, and* $\varphi : X \to Y$ *is a nonconstant morphism of nonsingular curves, then* φ *has only finitely many ramification points.*

CHAPTER 11

The Riemann-Roch and Riemann-Hurwitz theorems

We state without proof the Riemann-Roch theorem, and give some basic applications, including a proof of the Riemann-Hurwitz theorem. We use throughout the convention that all curves are projective and nonsingular.

11.1. The Riemann-Roch theorem

With all the preliminaries out of the way, we can state one of the most fundamental theorems in the study of algebraic curves, the Riemann-Roch theorem.

THEOREM 11.1.1. *Let X be a curve of genus g, and D a divisor on X of degree d. Then*

$$\ell(D) - \dim_k \Omega(-D) = d + 1 - g.$$

One immediate consequence is:

COROLLARY 11.1.2. *Let ω be a non-zero rational differential form on a curve X of genus g. Then $D(\omega)$ has degree $2g - 2$.*

PROOF. We apply the Riemann-Roch theorem in the case that $D = D(\omega)$. We know that $\Omega(-D)$ is isomorphic to $\mathcal{L}(D(\omega) - D)$, so $\Omega(D(\omega))$ is isomorphic to $\mathcal{L}(0) = k$. On the other hand, $\mathcal{L}(D(\omega))$ is isomorphic to $\Omega(0)$, so $\ell(D(\omega)) = g$. We conclude that

$$g - 1 = \deg D(\omega) + 1 - g,$$

giving the desired identity. □

However, the Riemann-Roch theorem has a wide range of applications. For instance, we see that every curve of genus 0 is isomorphic to $\mathbb{P}^1$.

EXERCISE 11.1.3. Let X be a curve. Suppose X has a divisor D of degree $d > 0$, such that $\ell(D) = d + 1$. Prove that $X \cong \mathbb{P}^1$. Conclude that if X has genus 0, then $X \cong \mathbb{P}^1$.
Hint: for the first part, begin with the case that $d = 1$.

The Riemann-Roch theorem also tells us that for divisors D of high enough degree, there is no difficulty in understanding $\ell(D)$:

COROLLARY 11.1.4. *Let X be a curve, and D a divisor of degree $d > 2g - 2$. Then $\ell(D) = d + 1 - g$.*

PROOF. If ω is a rational differential form, we know that $\Omega(-D) \cong \mathcal{L}(D(\omega) - D) = 0$, since $\deg D(\omega) - D < 0$. The statement then follows from the Riemann-Roch theorem. □

We can also use the Riemann-Roch theorem to study imbeddings of curves into projective spaces. For instance, using Corollary 11.1.4 and Corollary 9.4.5, we conclude:

COROLLARY 11.1.5. *If X is a projective nonsingular curve of genus g, and D is a divisor with $\deg D > 2g$, then the complete linear series $\mathcal{L}(D)$ induces a morphism $X \to \mathbb{P}^{\deg D - g}$ which is an isomorphism onto its image.*

Example 11.1.6. If X has genus 1, and D is any divisor of degree 3, then $\mathcal{L}(D)$ gives a morphism $X \to \mathbb{P}^2_k$ of degree 3 which is an isomorphism onto its image. Taking into account Exercise 9.3.19, this is the well-known fact that every curve of genus 1 can be realized as a curve in the plane of degree 3.

Exercise 11.1.7. Let X be a projective nonsingular curve, and K the divisor of a rational differential form on X. Show that the complete linear series $\mathcal{L}(K)$ is basepoint free, and that the associated morphism $\varphi : X \to \mathbb{P}^{g-1}$ defines an isomorphism of X onto its image if and only if there does not exist a morphism $X \to \mathbb{P}^1_k$ of degree 2.

We will see another direction of application of the Riemann-Roch theorem in the next section, where we will treat the Riemann-Hurwitz theorem.

Remark 11.1.8. The proof of the Riemann-Roch theorem is rather difficult, but we can at least explain why the statement is reasonable. We first observe that because $\ell(0) = 1$ and $\dim_k \Omega(0) = g$ by definition, the desired statement holds for $D = 0$. Now, given any divisor D and $P \in X$, we know that $\ell(D+P) - \ell(D)$ and $\dim_k \Omega(-D) - \dim_k \Omega(-D-P)$ are both equal to 0 or 1, and what we want to show is that one is equal to 1 if and only if the other is equal to 0 – the theorem then follows by induction on the number of points in D (counted in terms of the absolute value of their coefficients).

To see why this makes sense, let V_P be the one-dimensional vector space described as follows: if c is the coefficient of $[P]$ in D, set

$$V_P = (\{0\} \cup \{f \in K(X)^* : \operatorname{ord}_P(f) \geqslant -c-1\}) \, / \, (\{0\} \cup \{f \in K(X)^* : \operatorname{ord}_P(f) \geqslant -c\}) \, .$$

Then clearly we have an exact sequence

$$0 \to \mathcal{L}(D) \to \mathcal{L}(D+P) \to V_P,$$

and this is surjective on the right exactly when $\ell(D+P) - \ell(D) = 1$. On the other hand, if we take the dual of the natural injection $\Omega(-D-P) \hookrightarrow \Omega(-D)$, we obtain a surjection $\Omega(-D)^* \twoheadrightarrow \Omega(-D-P)^*$.

We will define a map $V_P \to \Omega(-D)^*$ using the idea of residues, for which we only need the simplest case: if ω is a rational differential form with at worst a simple pole at P, then we can write $\omega = g dt$, where t is a local coordinate at t and $\operatorname{ord}_P(g) \geqslant -1$. We then set $\operatorname{res}_P(\omega) = (tg)(P)$. In this case, it is very straightforward to check the following basic properties:

(i) $\operatorname{res}_P(\omega)$ doesn't depend on the choice of t;
(ii) $\operatorname{res}_P(\omega)$ is k-linear in ω;
(iii) $\operatorname{res}_P(\omega) = 0$ if and only if ω is in fact regular at P.

Now, given $f \in K(X)^*$ with $\operatorname{ord}_P(f) \geqslant -c-1$, and $\omega \in \Omega(-D)$ nonzero, we note that $f\omega$ has at worst a simple pole at P, so $\operatorname{res}_P(f\omega)$ gives an element of k. If in fact $\operatorname{ord}_P(f) \geqslant -c$, we get that $\operatorname{res}_P(f\omega) = 0$, so we see that we have constructed the desired k-linear map $V_P \to \Omega(-D)^*$. We see that the resulting sequence

$$V_P \to \Omega(-D)^* \to \Omega(-D-P)^* \to 0$$

is exact: indeed, it is clear that the image from the left is contained in the kernel, but both of these are either 0 or 1-dimensional, and we see that the map on the left is nonzero if and only if $\Omega(-D)$ contains some ω with order exactly c at P, if and only if the second map has nonzero kernel.

We thus have a sequence

$$0 \to \mathcal{L}(D) \to \mathcal{L}(D+P) \to V_P \to \Omega(-D)^* \to \Omega(-D-P)^* \to 0,$$

which is exact everywhere except possibly at V_P, and in order to prove the Riemann-Roch theorem, it is enough to prove exactness at V_P. Now, it is not hard to see why we have a complex at V_P: if we have $f \in \mathcal{L}(D+P)$ and $\omega \in \Omega(-D)$, then $f\omega$ is regular except possibly for a simple pole at P, and then the residue theorem (for algebraic curves) implies that $\mathrm{res}_P(f\omega) = 0$. The core of the proof of the Riemann-Roch theorem is then to prove that the image of the map to V_P contains the kernel of the map to $\Omega(-D)^*$.

11.2. The Riemann-Hurwitz theorem

As another application of the Riemann-Roch theorem, we prove the Riemann-Hurwitz theorem (also sometimes called the Hurwitz theorem), which relates the ramification of a separable morphism to its degree, as well as the genus of the curves. We give some basic applications, and also use it to prove the degree-genus formula in Exercise 11.2.6.

THEOREM 11.2.1. *Let $\varphi : X \to Y$ be a separable morphism of curves, of degree d. Let g_X and g_Y be the genus of X and Y respectively. Then we have*

$$2g_X - 2 \geqslant d(2g_Y - 2) + \sum_{P \in X} (e_P - 1),$$

with equality if and only if φ is tamely ramified.

More precisely, using Notation 10.3.7 we have

$$2g_X - 2 = 2(2g_Y - 2) + \sum_{P \in X} \mathrm{dord}_P\, \varphi \geqslant \sum_{P \in X} (e_P - 1),$$

with equality if and only if φ is tamely ramified.

PROOF. First, in the more precise statement, the inequality and criterion for equality is immediate from Proposition 10.3.5, so it suffices to prove the first equality. Let ω be a nonzero rational differential form on Y, such that $\varphi^*\omega$ is also nonzero; this exists by separability, using Theorem 10.3.3. By Corollary 11.1.2, we know that $D(\omega)$ has degree $2g_Y - 2$, and $D(\varphi^*\omega)$ has degree $2g_X - 2$. We claim that

$$D(\varphi^*\omega) = \varphi^* D(\omega) + \sum_{P \in X} (\mathrm{dord}_P\, \varphi)[P].$$

The desired result then follows by taking degrees, using Corollary 9.2.3 to conclude that $\deg \varphi^* D(\omega) = d \deg D(\omega)$. Given $P \in X$, let s be a local coordinate at P, and t be a local coordinate at $\varphi(P)$, and write $\omega = ft^n dt$ for some $f \in \mathscr{O}^*_{\varphi(P),Y}$, and n is by definition the coefficient of $[P]$ in $D(\omega)$. Then the coefficient of $[P]$ in $\varphi^*\omega = (\varphi^* ft^n)\varphi^* dt$ is its order of vanishing at P, which is the sum of $\mathrm{ord}_P\, \varphi^* ft^n$ and the order of vanishing of $\varphi^* dt$ at P. The latter is $\mathrm{dord}_P\, \varphi$ by definition, while we also have $\mathrm{ord}_P\, \varphi^* ft^n = n\, \mathrm{ord}_P\, \varphi^* t = e_P n$ by definition of the ramification index. The latter is also equal to the coefficient of $[P]$ in $\varphi^* D(ft^n)$, which in turn is by definition the coefficient of $[P]$ in $\varphi^* D(\omega)$. This proves the claim, and the theorem. □

One immediate consequence is the following:

COROLLARY 11.2.2. *Let $\varphi : X \to Y$ be a tamely ramified morphism of curves. Then $\sum_{P \in X}(e_P - 1)$ is even.*

EXAMPLE 11.2.3. Suppose k has characteristic p, and consider the morphism $\mathbb{P}^1 \to \mathbb{P}^1$ given on $\mathbb{A}^1$ by $x \mapsto x^{2p} - x$. Then one checks that this is unramified on $\mathbb{A}^1$, and ramified to order $2p$ at ∞; thus, Corollary 11.2.2 is false without the tame ramification hypothesis.

COROLLARY 11.2.4. *Let $\varphi : X \to Y$ be a separable morphism of curves of genus g_X and g_Y, respectively. Then $g_X \geqslant g_Y$, and equality is only possible if $g_Y = 0, 1$ or $d = 1$.*

PROOF. For $g_Y = 0$, there is nothing to prove. For $g_Y = 1$, the righthand side of the Riemann-Hurwitz formula is nonnegative, so $g_X \geqslant 1$. But for $g_Y > 1$, we have $g_X - 1 \geqslant d(g_Y - 1) \geqslant g_Y - 1$, with equality only possible if $d = 1$, so we conclude the desired statement. □

Remark 11.2.5. In fact, Corollary 11.2.4 holds for arbitrary nonconstant morphisms: one reduces to the separable case using that any such morphism factors as a composition of Frobenius morphisms followed by a separable morphism, and noting that the Frobenius morphisms leave the genus unchanged.

Exercise 11.2.6. Let $X = Z(F) \subseteq \mathbb{P}^2_k$ be a nonsingular plane curve, with $F \in k[X_0, X_1, X_2]$ irreducible of degree d. We will prove the **degree-genus formula**, that the genus g of X is given by

$$g = (d-1)(d-2)/2.$$

(a) First suppose that $(0,1,0) \notin X$. Let $\varphi : X \to \mathbb{P}^1_k$ be the morphism induced by projection from $(0,1,0)$. Show that for any $P \in X$, we have

$$\operatorname{dord}_P \varphi = i_P(X \cdot Z(\frac{\partial F}{\partial X_1})),$$

with notation as in Exercise 9.3.18.

(b) Using the Riemann-Hurwitz theorem and Exercise 9.3.20, prove the degree-genus formula.

Remark 11.2.7. Given our definition of genus, it is natural to wonder how easy it is to see that it agrees with the topological notion of genus in the case $k = \mathbb{C}$. One way to verify that these indeed agree is to prove the Riemann-Hurwitz theorem also for branched covers of surfaces, using the topological definition of genus, and then use that every curve has a nonconstant morphism to $\mathbb{P}^1_k$ to conclude that the definitions of genus must coincide.

11.3. Brill-Noether theory and applications

We conclude with a description of Brill-Noether theory and its application to the study of moduli spaces of curves. Although many of these ideas go back to the 19th century, many of the central results in the field were proved in the 1980's, with some questions remaining open.

A natural question to consider is the following:

Question 11.3.1. For a curve X, given $d, r \geqslant 0$, when does there exist a divisor D on X of degree d with $\ell(D) = r + 1$?

Because of the close connection between divisors and maps to projective space, its answer can be very important for understanding the geometry of X. In particular, it is very helpful to know how X may be imbedded into projective space on the one hand, and on the other what is the smallest degree of a nonconstant morphism $X \to \mathbb{P}^1_k$.

As stated, the question is too hard to answer for arbitrary curves X, but it turns out if we ask what happens on a "general" curve X of genus g, the problem becomes more approachable, and we have the following:

Theorem 11.3.2. *Given a triple (g, r, d) of nonnegative integers, every curve X of genus g has a divisor D of degree d with $\ell(D) \geqslant r + 1$ if and only if*

$$\rho := g - (r+1)(r+g-d) \geqslant 0.$$

In fact, one can make a parameter space $B^r_d(X)$ for such divisors, and the theorem further states that if $\rho \geqslant 0$, the minimal dimension of $B^r_d(X)$ as X varies over curves of genus g is equal to ρ.

This theorem is commonly known as the "Brill-Noether" theorem; it was stated in the 1870's by Brill and Noether, but it took a century before a full proof was provided, due to work of Castelnuouvo, Severi, Kempf, Kleiman, Laksov and Griffiths and Harris.

One of the striking applications of this theory was to the geometry of moduli spaces of curves. To explain this, we begin with some simple examples.

Any curve of genus 0 is isomorphic to $\mathbb{P}^1_k$, the projective line. Any curve of genus 1 can be realized as a plane cubic, and conversely every nonsingular plane cubic is a curve of genus 1. The genus-2 curves can all be written in the form $y^2 = f(x)$ where $f(x)$ is a polynomial of degree 5, without repeated roots. Curves of genus 3 break into two types: some can be written as in the genus-2 case, with $f(x)$ of degree 7 instead of 5, while the rest can be written as a curve of degree 4 in $\mathbb{P}^2_k$. The common thread among these examples is that we can write down all such curves explicitly. In fact, in each case, we can write down a single family of one or more polynomials with freely varying coefficients whose zero sets give all (or almost all, in a suitable sense) curves of the given genus. This can be expressed as saying that "the moduli space $\mathcal{M}_g$ parametrizing curves of genus g is unirational," a statement which we now explain.

A variety X is **unirational** if there exists a map from an open subset of $\mathbb{A}^n_k$ to X which has dense image. We will not be able to properly develop the theory of $\mathcal{M}_g$, but for now we only need to know a couple of its properties: first, it is a variety (or something of that nature) whose points correspond to curves of genus g, and second, given a family of curves of genus g, parametrizing by the points of a variety X, we obtain a morphism $X \to \mathcal{M}_g$ induced by, for each point of X, taking the corresponding curve in the family. Now, if we have a family of polynomials with freely varying coefficients, we can think of their zero sets as a family parametrized by $\mathbb{A}^n_k$, where n is the number of coefficients. In the cases above, we'll have to throw away some choices of coefficients if we want the zero sets to give nonsingular curves, but the 'bad' choices will be a closed subset, so we still obtain a family of curves parametrized by an open subset of $\mathbb{A}^n_k$, and as a result, we get a map from an open subset of $\mathbb{A}^n_k$ to $\mathcal{M}_g$. The above examples show that $\mathcal{M}_g$ is unirational for $g \leqslant 3$.

It is then natural to ask:

QUESTION 11.3.3. Is $\mathcal{M}_g$ unirational for all g?

Famously, Severi erroneously claimed an affirmative answer to this question, but work of Harris, Mumford and Eisenbud in the 1980's proved that in fact, $\mathcal{M}_g$ is not unirational for $g \geqslant 23$. In practice, this means that it is not possible to explicitly write down a "general" curve of large genus. Remarkably, their work was based on Brill-Noether theory, as follows: it turns out that the main thing one has to do is to write down effective divisors on $\mathcal{M}_g$ itself, and carry out some computations regarding these divisors. One of the two classes of divisors considered by Eisenbud and Harris were obtained by choosing r and d so that $\rho = -1$, and consisting the collection of curves of genus g having a linear series of degree d and rank r; the second class was similar, but slightly more complicated.

There have been further results since then, but it is not yet settled precisely which values of g have $\mathcal{M}_g$ unirational, and this (along with refined variants of the same question) remains an active subject of research.

APPENDIX A

Complex varieties and the analytic topology

Classical algebraic geometers studied algebraic varieties over the complex numbers. In this setting, they didn't have to worry about the Zariski topology and its many pathologies, because they already had a better-behaved topology to work with: the analytic topology inherited from the usual topology on the complex numbers themselves. In this appendix, we introduce the analytic topology, and explore some of its basic properties. We also investigate how it interacts with properties of varieties which we have already defined. The definitions would go through just as well without the irreducibility hypothesis, but since we have developed abstract (pre)varieties assuming irreducibility, we will restrict ourselves to that context.

A.1. Affine complex varieties

An affine variety $X \subseteq \mathbb{A}^n_{\mathbb{C}}$ over the complex numbers is the zero set of a system of polynomials in n variables. Unlike the Zariski topology, the analytic topology on X corresponds to our topological intuition for what X "looks like." We define:

Definition A.1.1. The **analytic topology** on X is the topology induced by the inclusion $X \hookrightarrow \mathbb{A}^n_{\mathbb{C}} \cong \mathbb{C}^n$, using the usual topology on $\mathbb{C}^n$. The topological space of X endowed with the analytic topology is denoted by X_{an}.

We will continue to use X to denote the variety together with its Zariski topology. Because zero sets of (multivariate) polynomials are closed in $\mathbb{C}^n$, the analytic topology is finer than the Zariski topology: that is, a closed subset in the Zariski topology is closed in the analytic topology, but not in general vice versa. This may be rephrased into the following conclusion:

Proposition A.1.2. *The map of topological spaces $X_{\text{an}} \to X$ induced by the identity on points is continuous.*

Similar arguments also prove the following basic facts.

Exercise A.1.3. (a) A regular function $X \to \mathbb{C}$ gives a continuous map $X_{\text{an}} \to \mathbb{C}$, where $\mathbb{C}$ is equipped with the analytic topology.

(b) If also $Y \subseteq \mathbb{A}^m_{\mathbb{C}}$ is an affine complex variety, and $X \to Y$ is a morphism, then the induced map $X_{\text{an}} \to Y_{\text{an}}$ is continuous.

(c) The analytic topology on X is an isomorphism invariant, independent of the particular imbedding of X into affine space.

We will need one elementary result on the continuity of roots of a single-variable complex polynomial.

Theorem A.1.4. *Let $f(x) = a_0 + a_1x + \cdots + a_dx^d$ be a nonzero complex polynomial, and let $c \in \mathbb{C}$ be a root of f. Then for any $\epsilon > 0$, there exists $\delta > 0$ such that for any $b_0, \ldots, b_d \in \mathbb{C}$ with $|a_i - b_i| < \delta$ for all i, there is a root c' of the polynomial $g(x) = b_0 + b_1x + \cdots + b_dx^d$ with $|c - c'| < \epsilon$.*

Proof. Let γ be a circle of radius less than ϵ around c, chosen so that there are no other zeros of $f(x)$ inside (or on) γ. Let w be the minimum value of $|f(x)|$ on γ, which is strictly positive by hypothesis. For δ sufficiently small, we have that if $b_0, \ldots, b_d \in \mathbb{C}$ satisfy $|a_i - b_i| < \delta$, then $|f(x) - g(x)| < w$ for all $x \in \gamma$, where $g(x) = b_0 + b_1 x + \cdots + b_d x^d$. It follows from Rouche's theorem that f and g have the same number of roots inside the circle γ, which gives the desired statement. $\square$

We can now begin to make statements about the analytic topology of varieties, at least in some special cases.

Corollary A.1.5. *Let $C \subseteq \mathbb{A}^2_{\mathbb{C}}$ be a curve in the affine plane. Then C_{an} has no isolated points.*

Proof. Since C has codimension 1 in $\mathbb{A}^2_{\mathbb{C}}$, we know it can be expressed as the zero set of a single polynomial, say $f(x, y) \in \mathbb{C}[x, y]$, with $\deg f = d$. Then $f(x, y)$ has degree at most d when considered as a polynomial in y, and its coefficients are themselves continuous functions of x. We may assume that $f(x, y)$ is not constant in y, since otherwise by irreducibility of C we must have $f = x - c$ for some c, so C is a vertical line and certainly has no isolated points. Thus, given $(x_0, y_0) \in C$, it follows from Theorem A.1.4 that for any $\epsilon > 0$, there is some $\delta > 0$ such that for every x with $|x - x_0| < \delta$, there is some y with $|y - y_0| < \epsilon$ and $f(x, y) = 0$. We conclude that (x_0, y_0) is not an isolated point of C. $\square$

A.2. The analytic topology on prevarieties

Having defined the analytic topology on affine varieties, we can now define it on prevarieties, since they are glued together from affine varieties.

Definition A.2.1. Let X be a prevariety with atlas $\{\varphi_i : X_i \to U_i\}$. The **analytic topology** on X, denoted by X_{an}, is the topology such that for each i, the map $(X_i)_{\mathrm{an}} \to (X_{\mathrm{an}})|_{U_i}$ induced by φ_i is a homeomorphism.

Once again, there are some basic properties to check for the analytic topology:

Exercise A.2.2. (a) The analytic topology X_{an} on a prevariety X is well defined.
(b) For any $U \subseteq X$ an affine open subset, $X_{\mathrm{an}}|_U = U_{\mathrm{an}}$.
(c) The map of topological spaces $X_{\mathrm{an}} \to X$ induced by the identity on points is continuous.
(d) Given another prevariety Y with atlas $\{\psi_i : Y_i \to V_i\}$, and a morphism $\varphi : X \to Y$, the induced map $X_{\mathrm{an}} \to Y_{\mathrm{an}}$ is continuous.
(e) The analytic topology on a prevariety X is an isomorphism invariant.
(f) If $Z \subseteq X$ is a subprevariety, then Z_{an} has the subspace topology inside X_{an}.

Example A.2.3. Complex projective space $\mathbb{P}^n_{\mathbb{C}}$ is compact in the analytic topology. Indeed, we have the morphism $\mathbb{A}^{n+1}_{\mathbb{C}} \smallsetminus \{0\} \to \mathbb{P}^n_{\mathbb{C}}$ defined by $(x_0, \ldots, x_n) \mapsto (x_0, \ldots, x_n)$, which by Exercise A.2.2 (d) induces a continuous map $(\mathbb{A}^{n+1}_{\mathbb{C}} \smallsetminus \{0\})_{\mathrm{an}} \to (\mathbb{P}^n_{\mathbb{C}})_{\mathrm{an}}$ Inside $(\mathbb{A}^{n+1}_{\mathbb{C}} \smallsetminus \{0\})_{\mathrm{an}}$ we have the closed subset consisting of elements of norm 1; identifying $\mathbb{A}^{n+1}_{\mathbb{C}}$ with $\mathbb{R}^{2n+2}$, this subset is precisely the unit sphere, and hence compact. Moreover, it surjects onto $(\mathbb{P}^n_{\mathbb{C}})_{\mathrm{an}}$, and since the continuous image of a compact set is compact, we conclude that $(\mathbb{P}^n_{\mathbb{C}})_{\mathrm{an}}$ is compact.

As an immediate consequence of Example A.2.3 and Exercise A.2.2, we conclude:

Corollary A.2.4. *If X is a projective variety, then X_{an} is compact.*

We know that if X, Y are prevarieties, the Zariski topology on $X \times Y$ is not the product topology. However, (again confirming that the analytic topology behaves closer to our intuition) for the analytic topology we have:

Exercise A.2.5. If X, Y are prevarieties, then

$$(X \times Y)_{\text{an}} = (X_{\text{an}}) \times (Y_{\text{an}}),$$

where the righthand side denotes the product topology of the analytic topologies on X and Y.

We then conclude:

Corollary A.2.6. *If X is a variety, then X_{an} is Hausdorff.*

Proof. For X_{an} to be Hausdorff is equivalent to the image $\Delta(X_{\text{an}})$ in $X_{\text{an}} \times X_{\text{an}}$ to be closed. Since X is a variety, $\Delta(X)$ is closed in $X \times X$ in the Zariski topology, so $\Delta(X_{\text{an}})$ is closed in $(X \times X)_{\text{an}}$ by Exercise A.2.2 (c). But $(X \times X)_{\text{an}} = X_{\text{an}} \times X_{\text{an}}$ by Exercise A.2.5, so we conclude that X_{an} is Hausdorff by Exercise 5.3.1. $\square$

In fact, we will see in Corollary A.3.3 that the converse also holds: if X is a complex prevariety and X_{an} is Hausdorff, then X is a variety. However, this requires putting together some of the deeper results which we have developed.

We can then generalize Corollary A.1.5 as follows:

Corollary A.2.7. *Let X be a complex one-dimensional prevariety. Then X_{an} has no isolated points.*

Proof. We prove the statement in several steps. We first prove it for projective nonsingular curves $X \subseteq \mathbb{P}^n_{\mathbb{C}}$. Given $P \in X$, we know by Proposition 9.5.1 that there exists a morphism $\varphi : X \to \mathbb{P}^2_{\mathbb{C}}$ such that $\varphi^{-1}(\varphi(P)) = \{P\}$. Then $\varphi(X)$ is a projective (possibly singular) curve. If P were isolated in X_{an}, then its complement would be a closed subset, hence compact by Corollary A.2.4. Then, by Corollary A.2.6 we would have $\varphi(X \smallsetminus P) = \varphi(X) \smallsetminus \varphi(P)$ is closed in $\varphi(X)$, so that $\varphi(P)$ is isolated in $\varphi(X)_{\text{an}}$. But if $U \cong \mathbb{A}^2_{\mathbb{C}}$ is an open neighborhood of $\varphi(P)$, then $\varphi(X) \cap U$ is an affine plane curve, and by Corollary A.1.5 we conclude that $\varphi(P)$ is not an isolated point of $(\varphi(X) \cap U)_{\text{an}}$. We conclude that P could not have been isolated in X_{an}.

The case of an arbitrary nonsingular curve X then follows, since we know that X can be realized as a Zariski open subset of some nonsingular projective $\bar{X}$, and $\bar{X} \smallsetminus X$ consists of finitely many points, so if $\bar{X}_{\text{an}}$ has no isolated points then X_{an} cannot have any isolated points either. Note that an alternate proof of the statement for nonsingular curves is given by Corollary A.4.1 below.

We conclude the statement of the corollary for an arbitrary curve X by considering the normalization morphism $\nu : \widetilde{X} \to X$; this is a surjective morphism, with $\widetilde{X}$ a nonsingular curve. For any $P \in X$, we have $\nu^{-1}(P)$ a closed set, hence a finite set of points, which cannot be open in $\widetilde{X}_{\text{an}}$ since we know it has no isolated points. Thus $\{P\}$ is not an open subset of X_{an}, so X_{an} does not have any isolated points.

Finally, if X is any one-dimensional prevariety, every point has an affine open neighborhood which is a curve, so applying the previous case we conclude that X_{an} has no isolated points. $\square$

A.3. Fundamental results

There are a number of basic and important facts relating the ideas we have introduced for varieties to standard topological properties applied to the analytic topology. The main ingredient for proving these statements is the following:

Theorem A.3.1. *Let X be a prevariety, and U a Zariski open subset. Then U is dense in X_{an}.*

Proof. We first observe that the statement of the theorem in the case that X is a curve is precisely Corollary A.2.7. Now, for any X, given $P \in X \smallsetminus U$, by Lemma 8.3.5 there exists a curve Z in X with $P \in Z$ and $Z \cap U \neq \emptyset$. By Exercise A.2.2 (f), we have $Z_{\text{an}} = (X_{\text{an}})|_Z$, and P is in the (analytic) closure of $Z_{\mathrm{an}} \cap U$ since we already that the result hold for curves. It thus follows that P is in the closure of U in X_{an}, and we conclude that U is dense, as asserted. $\square$

The theorem is particularly powerful in combination with Chevalley's theorem, which together yield:

COROLLARY A.3.2. *Let $\varphi : X \to Y$ be a morphism of prevarieties, and suppose $\varphi(X)$ is closed in the analytic topology. Then $\varphi(X)$ is closed in the Zariski topology.*

PROOF. Let Z be the Zariski closure of $\varphi(X)$ in Y. By Theorem 8.1.2, we have that $\varphi(X)$ contains a Zariski open subset U of Z. By Theorem A.3.1, we have that U is dense in Z_{an}, so we conclude that $\varphi(X) = Z$, and $\varphi(X)$ is Zariski-closed, as desired. $\square$

We can now prove two basic statements on the analytic topology.

COROLLARY A.3.3. *A complex prevariety X is a variety if and only if X_{an} is Hausdorff.*

PROOF. One direction was proved already in Corollary A.2.6. Conversely, suppose X_{an} is Hausdorff, so that the diagonal $\Delta(X)$ is closed in the analytic topology. Then by Corollary A.3.2, we have $\Delta(X)$ closed also in the Zariski topology, so X is a variety. $\square$

COROLLARY A.3.4. *A complex variety X is complete if and only if X_{an} is compact.*

PROOF. First suppose that X_{an} is compact. We wish to show that for every prevariety Y, the projection morphism $p_2 : X \times Y \to Y$ is closed (in the Zariski topology). Since X_{an} is compact, Exercise 8.2.1 implies that $X_{\mathrm{an}} \times Y_{\mathrm{an}} \to Y_{\mathrm{an}}$ is closed, and by Exercise A.2.5 $X_{\mathrm{an}} \times Y_{\mathrm{an}} = (X \times Y)_{\mathrm{an}}$. Let $Z \subseteq X \times Y$ be Zariski closed. Then it is also closed in $(X \times Y)_{\mathrm{an}}$, so $p_2(Z)$ in Y_{an} is closed. But it follows from Corollary A.3.2 that $p_2(Z)$ is closed in Y the Zariski topology, which means that X is complete.

For the converse, first suppose that X is projective. Then X_{an} is closed subset of $(\mathbb{P}^n_{\mathbb{C}})_{\mathrm{an}}$, and is thus compact by Example A.2.3. Now suppose that X is complete. Then by Chow's Lemma (Exercise 8.3.8), there is a surjective morphism $X' \to X$ for some projective variety X'. Thus, X_{an} is the continuous image of the compact space $(X')_{\mathrm{an}}$, and is therefore compact. $\square$

A.4. Nonsingularity and complex manifolds

In order to conclude our study of the complex topology, we will need to know an important and basic fact: a nonsingular complex curve has the natural structure of a (one-dimensional) complex manifold. We will prove the more general statement for complex varieties of any dimension, as a corollary of the Jacobian criterion.

COROLLARY A.4.1. *Let X be a complex prevariety, and $P \in X$ a nonsingular point. If $(t_1, \dots, t_d)$ is a system of local coordinates for X at P, there exists a neighborhood U of P in X_{an} on which all the t_i induce sections of $\mathscr{O}_X$, and such that the induced map $U \to \mathbb{C}^d$ defines a homeomorphism of U_{an} onto an open neighborhood V of the origin in $\mathbb{C}^d$, with every element of $\mathscr{O}_X$ corresponding to an analytic function on the image of its domain of definition in V.*

In particular, if X is a nonsingular variety of dimension d, then X_{an} has the structure of a complex manifold of dimension d, with sections of $\mathscr{O}_X$ giving rise to analytic functions on X_{an}.

PROOF. The first statement being local on X, we may assume that X is affine and the t_i are global sections of $\mathscr{O}_X$. Then for some n we can extend the t_i to a set of n generators of $A(X)$ over $\mathbb{C}$, and let I be the corresponding ideal of definition for X in $\mathbb{A}^n_{\mathbb{C}}$. Reindexing and applying Corollary 4.3.2 and Exercise 4.3.6, we conclude that I is generated in a neighborhood of P by some $f_1, \dots, f_{n-d}$, and that $\left(\frac{\partial f_i}{\partial t_j}\right)_{1 \leqslant i,j \leqslant n-d}$ is invertible at P. By the implicit function theorem, there exist neighborhoods U of P in X_{an}, V of the origin in $\mathbb{C}^d$ and an analytic function $g : V \to \mathbb{C}^{n-d}$ such that $\mathrm{id} \times g : V \to \mathbb{C}^n$ maps onto U and is inverse to the projection map $\mathbb{C}^n \to \mathbb{C}^d$. This gives the desired homeomorphism. In addition, g expresses all the t_i as analytic functions of $t_1, \dots, t_d$, and since sections of $\mathscr{O}_X$ are expressed locally as rational functions in the t_i with denominators nonvanishing on their domain of definition, we conclude that they are analytic on their domain of definitions.

For the final statement, if X is a variety X_{an} is Hausdorff by Corollary A.2.6. The first statement gives us a complex atlas on an open cover of X_{an}, and to conclude that the induced transition maps are analytic, we need only use that algebraic isomorphisms are analytic under the above correspondence, which follows easily from the fact that sections of $\mathscr{O}_X$ yield analytic functions. □

REMARK A.4.2. Recall that every complex manifold is naturally oriented. Indeed, $\mathbb{C}^d$ inherits an orientation from $\mathbb{C}$, and this orientation is necessarily preserved under holomorphic transition maps, so local orientations from charts induce global orientation on the manifold.

A.5. Connectedness

We conclude our discussion of the complex topology with the following foundational theorem:

THEOREM A.5.1. *Let X be a prevariety over $\mathbb{C}$. Then X is connected in the Zariski topology.*

As we shall see, this theorem is quite a bit deeper than the corresponding statements for properness and separatedness. This difficulty should perhaps not be surprising: if we consider the analogous statement over $\mathbb{R}$, it remains true that a variety is Hausdorff in the real analytic topology, and a complete variety is compact, but it is not true that a variety is connected in the real analytic topology, as we see already with hyperbolas in the affine plane, or elliptic curves in the projective plane. Nonetheless, with the tools we have developed the proof of the theorem will not be very difficult.

PROOF. We begin by proving that the desired result holds in the case that X is a nonsingular projective curve. Given $P \in X$, the Riemann-Roch theorem implies that there exists a (nonconstant) rational function f with a pole only at P. We know by Corollary A.3.4 and Corollary A.4.1 that X_{an} is a compact one-dimensional complex manifold, and f (being a quotient of analytic functions) induces a meromorphic function on X_{an}. If X_{an} is disconnected, let $C \subseteq X_{\mathrm{an}}$ be a connected component not containing P. Then f is analytic on C, so the maximum modulus principle implies that f is constant on C. But then subtracting this constant we obtain a non-zero rational function with infinitely many zeroes on X, which is impossible. Thus, X_{an} is connected.

Next, if X is an arbitrary nonsingular curve, we know that it may be imbedded as an open subset of a nonsingular projective curve $\overline{X}$. We now know that $\overline{X}_{\mathrm{an}}$ is a connected complex manifold of dimension 1. But the complement of X_{an} in $\overline{X}_{\mathrm{an}}$ is a finite set of points, so we conclude that X_{an} is likewise connected, as desired.

Now suppose that X is any curve, and let $\widetilde{X} \to X$ be the normalization. Then we have shown that $\widetilde{X}_{\mathrm{an}}$ is connected, but $\widetilde{X}_{\mathrm{an}}$ surjects onto X_{an}, so we conclude that X_{an} is likewise connected.

Finally, if X is any one-dimensional prevariety, and $U \subseteq X$ a nonempty affine open subset, then U is a curve, so U_{an} is connected. On the other hand, by Corollary A.2.7, U_{an} is dense in X_{an}, so we conclude that X_{an} is likewise connected.

For the higher-dimensional case, suppose that X is arbitrary. By Theorem 8.4.10, any two points of X are connected by a connected chain of closed one-dimensional subprevarieties. Since we know each of the latter is connected in the analytic topology, it follows that X_{an} is connected, as desired. □

Remark A.5.2. Under our running irreducibility hypothesis, X is always connected. However, if we had developed the analytic topology without any irreducibility hypothesis, then we could prove with much the same argument that in fact X is connected if and only if X_{an} is connected.

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