Chapter 5
Krull Dimension and Transcendence Degree

In this chapter, we introduce the Krull dimension, which is the "correct" concept of dimension in algebraic geometry and commutative algebra. Then we prove that the dimension of an affine algebra is equal to its transcendence degree. This makes the dimension more accessible both to computation and to interpretation.

We start by introducing the following ad hoc notation. Let $\mathcal{M}$ be a set whose elements are sets. By a **chain** in $\mathcal{M}$ we mean a subset $\mathcal{C} \subseteq \mathcal{M}$ that is totally ordered by inclusion "$\subseteq$". The length of $\mathcal{C}$ is defined to be $\operatorname{length}(\mathcal{C}) := |\mathcal{C}| - 1 \in \mathbb{N}_0 \cup \{-1, \infty\}$. A finite chain of length n is usually written as

$$X_0 \subsetneqq X_1 \subsetneqq \cdots \subsetneqq X_n.$$

We write

$$\operatorname{length}(\mathcal{M}) := \sup \{\operatorname{length}(\mathcal{C}) \mid \mathcal{C} \text{ is a chain in } \mathcal{M}\} \in \mathbb{N}_0 \cup \{-1, \infty\}$$

(the length -1 occurs if $\mathcal{M} = \emptyset$).

Observe that the dimension of a vector space V is the maximal length of a chain of subspaces, i.e.,

$$\dim(V) = \operatorname{length}\Big(\{U \subseteq V \mid U \text{ subspace}\}\Big).$$

With this in mind, the following definition does not appear too far-fetched.

Definition 5.1 (Krull dimension)**.**

(a) Let X be a topological space. Set $\mathcal{M}$ to be the set of all closed, irreducible subsets of X. Then the **dimension** *of X (also called the* **Krull dimension***) is defined as*

$$\dim(X) := \operatorname{length}(\mathcal{M}).$$

(b) Let R be a ring. Then the **dimension** *of R (also called the* **Krull dimension***) is defined as*

G. Kemper, *A Course in Commutative Algebra*, Graduate Texts in Mathematics 256, DOI 10.1007/978-3-642-03545-6_6,

$$\dim(R) := \dim\left(\operatorname{Spec}(R)\right).$$

So $\dim(R) = \operatorname{length}\left(\operatorname{Spec}(R)\right)$*, since the closed, irreducible subsets of* $\operatorname{Spec}(R)$ *correspond to prime ideals of* R *by Proposition 3.6*(e) *and Theorem 3.10*(b)*. In other words, the dimension of* R *is the maximal length of a chain of prime ideals of* R*.*

(c) Let K *be a field. The dimension of a subset* $X \subseteq K^n$ *is the dimension of* X *with the Zariski topology. So if* K *is algebraically closed and* X *is an affine variety, then*

$$\dim(X) = \dim\left(K[X]\right),$$

since the closed, irreducible subsets of X *correspond to prime ideals in the coordinate ring* $K[X]$ *by Theorem 1.23 and Theorem 3.10*(a)*.*

Example 5.2. (1) If $X = \{P\}$ is a singleton or (more generally) a nonempty, finite, discrete topological space, then $\dim(X) = 0$. Moreover, $\dim(\emptyset) = -1$.

(2) If K is an infinite field and $X = K^1$, then the closed, irreducible subsets are the singletons and all of X, so $\dim(K^1) = 1$.

(3) Let $X \subseteq \mathbb{R}^3$ be the union of a plane P and a line L that is not contained in the plane. We can see two types of nonrefinable chains of closed, irreducible subsets:

(a) A point of L not lying in P, followed by all of L

(b) A point of P, followed by a line in P that contains the point, followed by all of P

From this, we see that $\dim(X) \geq 2$. Intuition tells us that the dimension should be equal to 2, but we cannot verify this yet.

(4) Every field has Krull dimension 0.

(5) The ring $\mathbb{Z}$ of integers has Krull dimension 1, with all maximal chains of prime ideals of the form $\{0\} \subsetneqq (p)$ with p a prime number.
More generally, every principal ideal domain that is not a field has Krull dimension 1.

(6) In particular, a polynomial ring $K[x]$ over a field has $\dim\left(K[x]\right) = 1$.

(7) Let K be a field and $R = K[x_1, x_2, \ldots]$ a polynomial ring in countably many indeterminates x_i, $i \in \mathbb{N}$. Then $P_i = (x_1, \ldots, x_i)$ provides an infinite chain of prime ideals, so $\dim(R) = \infty$. ◁

Remark. The ring in Example 5.2(7) is not Noetherian. It is tempting to hope that Noetherian rings are always finite-dimensional. However, Exercise 7.7 dashes this hope. The converse is also not true: Combining Example 2.3 and Exercise 5.3 yields a non-Noetherian integral domain of Krull dimension 2. ◁

Remark. If X is a Noetherian topological space with irreducible components $Z_1, \ldots, Z_n$, then

$$\dim(X) = \max\left\{\dim(Z_1), \ldots, \dim(Z_n), -1\right\}.$$

This follows from Theorem 3.11(b). We will call X **equidimensional** if all Z_i have the same dimension. Likewise, a Noetherian ring R is called **equidimensional** if $\operatorname{Spec}(R)$ is equidimensional. ◁

As we see from Example 5.2, it is very difficult to apply Definition 5.1 directly for determining the dimension of a variety. At this point we are not even able to determine the dimension of K^n (or of the polynomial ring $K[x_1, \ldots, x_n]$), although we easily get n as a lower bound. Another disadvantage is that at this point it is far from clear that the Krull dimension of an affine variety coincides with what we intuitively understand by dimension. The main result of this chapter is an "alternative definition" of the dimension of an affine algebra, which is much more accessible and more intuitive (see Remark 5.4). Another, less well-known, alternative definition, which holds for general rings, is given in Exercise 6.8.

Recall that a subset $\{a_1, \ldots, a_n\} \subseteq A$ of size n of an algebra A over a field K is called algebraically independent if for all nonzero polynomials $f \in K[x_1, \ldots, x_n]$ we have $f(a_1, \ldots, a_n) \neq 0$.

Definition 5.3. *Let A be an algebra over a field K. Then the* **transcendence degree** *of A is defined as*

$$\operatorname{trdeg}(A) := \sup \{|T| \mid T \subseteq A \text{ is finite and algebraically independent}\}.$$

So $\operatorname{trdeg}(A) \in \mathbb{N}_0 \cup \{-1, \infty\}$, *where* -1 *occurs if* $A = \{0\}$ *is the zero ring. (We set* $\sup \emptyset := -1$.*)*

Our next goal is to show that the dimension and the transcendence degree of an affine algebra coincide. The following remark is intended to convince the reader that this is a worthy goal.

Remark 5.4. (a) Let $A = K[X]$ be the coordinate ring of an affine variety over an infinite field. Finding an algebraically independent subset of size n of A is equivalent to finding an injective homomorphism $K[x_1, \ldots, x_n] \to A$. By Exercise 4.1(a), this is the same as giving a dominant morphism $X \to K^n$. So $\operatorname{trdeg}(A)$ is the largest number n such that there exists a dominant morphism $X \to K^n$. This already links the transcendence degree to an intuitive concept of dimension. In fact, we will be able to do even better: In Chapter 8, we will see that such a morphism can be chosen to be surjective, and such that every point in K^n has only finitely many preimages (see after Remark 8.20 on page 105).

(b) If $A = K[x_1, \ldots, x_n]/I$ is an affine algebra given by generators of an ideal $I \subseteq K[x_1, \ldots, x_n]$, then $\operatorname{trdeg}(A)$ can be computed algorithmically by Gröbner basis methods. We will see this in Chapter 9 (see on page 128). So equating dimension and transcendence degree brings the dimension into the realm of computability. ◁

Theorem 5.5 (Dimension of algebras, upper bound)**.** *Let A be a (not necessarily finitely generated) algebra over a field K. Then*

$$\dim(A) \le \operatorname{trdeg}(A).$$

Proof. This is the special case $S = A$ of the following lemma. □

Lemma 5.6. *Let A be an algebra over a field K, and let $S \subseteq A$ be a subset that generates A as an algebra. Then*

$$\dim(A) \le \sup\left\{|T| \mid T \subseteq S \text{ is finite and algebraically independent}\right\}.$$

Proof. Let n be the supremum on the right-hand side of the claimed inequality. There is nothing to show if $n = \infty$, and the lemma is correct if $n = -1$. So assume $n \in \mathbb{N}_0$. We need to show that $\dim(A/P) \le n$ for all $P \in \operatorname{Spec}(A)$. If we substitute A by A/P and S by $\{a + P \mid a \in S\}$, then n cannot increase. Therefore we may assume that A is an integral domain.

First consider the case $n = 0$. Then all elements from S are algebraic, so the field of fractions $\operatorname{Quot}(A)$ is generated as a field extension of K by algebraic elements. It follows that $\operatorname{Quot}(A)$ is algebraic, so A is algebraic, too. By Lemma 1.1(a), this implies that A is a field, so $\dim(A) = 0$.

Now assume $n > 0$, and let

$$P_0 \subsetneqq P_1 \subsetneqq \cdots \subsetneqq P_m$$

be a chain in $\operatorname{Spec}(A)$ of length $m > 0$. Factoring by P_1 yields a chain in $\operatorname{Spec}(A/P_1)$ of length $m - 1$ (see Lemma 1.22). If we can show that all algebraically independent subsets $T \subseteq \{a + P_1 \mid a \in S\} \subseteq A/P_1$ have size $|T| < n$, then we can use induction on n and conclude that $\dim(A/P_1) < n$, so $m - 1 < n$, which yields the lemma.

By way of contradiction, assume that there exist $a_1, \ldots, a_n \in S$ such that $\{a_1 + P_1, \ldots, a_n + P_1\} \subseteq A/P_1$ is algebraically independent of size n. Then also $\{a_1, \ldots, a_n\} \subseteq S$ is algebraically independent. By the definition of n, all $a \in S$ are algebraic over $L := \operatorname{Quot}\left(K[a_1, \ldots, a_n]\right)$, so $\operatorname{Quot}(A)$ is algebraic over L, too. There exists a nonzero element $a \in P_1$. We have a nonzero polynomial $G = \sum_{i=0}^{k} g_i x^i \in L[x]$ with $G(a) = 0$. Since $a \neq 0$, we may assume $g_0 \neq 0$. Furthermore, we may assume $g_i \in K[a_1, \ldots, a_n]$. Then

$$g_0 = -\sum_{i=1}^{k} g_i a^i \in P_1,$$

so viewing g_0 as a polynomial in n indeterminates over K, we obtain $g_0(a_1 + P_1, \ldots, a_n + P_1) = 0$, contradicting the algebraic independence of the $a_i + P_1 \in A/P_1$. This completes the proof. □

We can now determine the dimension of polynomial rings over fields and of affine n-space K^n.

Corollary 5.7 (Dimension of a polynomial ring)**.** *If K is a field, then*

$$\dim\left(K[x_1,\ldots,x_n]\right) = n.$$

Moreover,

$$\dim\left(K^n\right) = \begin{cases} n & \text{if } K \text{ is infinite,} \\ 0 & \text{if } K \text{ is finite.} \end{cases}$$

Proof. With $S := \{x_1,\ldots,x_n\}$, Lemma 5.6 yields $\dim\left(K[x_1,\ldots,x_n]\right) \leq n$. Since we have the chain

$$\{0\} \subsetneqq (x_1) \subsetneqq (x_1,x_2) \subsetneqq \cdots \subsetneqq (x_1,\ldots,x_n) \tag{5.1}$$

of length n in $\operatorname{Spec}\left(K[x_1,\ldots,x_n]\right)$, equality holds.

Moreover, a chain of length m of closed, irreducible subsets $X_i \subseteq K^n$ gives rise to a chain of length m of ideals $\mathcal{I}(X_i) \subset K[x_1,\ldots,x_n]$, which are prime by Theorem 3.10(a), so $m \leq n$ by the above. On the other hand, if K is infinite, the affine varieties corresponding to the ideals in (5.1) are irreducible and provide a chain of length n, so $\dim\left(K^n\right) = n$. If K is a finite field, then $\dim\left(K^n\right) = 0$ by Example 5.2(1). □

Example 5.8. The bound from Theorem 5.5 is not always sharp. Indeed, consider the rational function field $A = K(x_1,\ldots,x_n) := \operatorname{Quot}\left(K[x_1,\ldots,x_n]\right)$. We have

$$\dim(A) = 0 < n = \operatorname{trdeg}(A).$$

◁

In Chapter 7 we will prove that if $R \neq \{0\}$ is a Noetherian ring, then

$$\dim\left(R[x]\right) = \dim(R) + 1$$

(Corollary 7.13), generalizing Corollary 5.7. In Exercise 7.10, the analogous result will be proved for the formal power series ring $R[[x]]$. For the formal power series ring in n indeterminates over a field K, this implies

$$\dim\left(K[[x_1,\ldots,x_n]]\right) = n.$$

Theorem 5.9 (Dimension and transcendence degree)**.** *Let A be an affine algebra. Then*

$$\dim(A) = \operatorname{trdeg}(A).$$

We will prove the theorem together with the following proposition, which often facilitates the computation of the transcendence degree since the set S can be taken to be finite.

Proposition 5.10 (Calculating the transcendence degree)**.** *Let A be an affine algebra, and let $S \subseteq A$ be a generating set. Then*

$$\operatorname{trdeg}(A) = \sup\left\{|T| \mid T \subseteq S \text{ is finite and algebraically independent}\right\}.$$

Proof of Theorem 5.9 and Proposition 5.10. By Lemma 5.6 we have

$$\dim(A) \leq \sup\{|T| \mid T \subseteq S \text{ is finite and algebraically independent}\},$$

and this supremum is clearly less than or equal to $\operatorname{trdeg}(A)$. So we need to show only that $\operatorname{trdeg}(A) \leq \dim(A)$. Using induction on n, we will show that if $\operatorname{trdeg}(A) \geq n$, then $\dim(A) \geq n$. We may assume $n > 0$. So let $a_1, \ldots, a_n \in A$ be algebraically independent. By Corollary 2.12, A is Noetherian, so by Corollary 3.14(a), there exist only finitely many minimal prime ideals $M_1, \ldots, M_r$ of A. Assume that for all $i \in \{1, \ldots, r\}$ we have that $a_1 + M_i, \ldots, a_n + M_i \in A/M_i$ are algebraically dependent. Then there exist polynomials $f_i \in K[x_1, \ldots, x_n] \setminus \{0\}$ such that $f_i(a_1, \ldots, a_n) \in M_i$, so

$$a := \prod_{i=1}^{r} f_i(a_1, \ldots, a_n) \in \bigcap_{i=1}^{r} M_i = \operatorname{nil}(A),$$

where the last equality follows from Corollary 3.14(c). So there exists a k with $a^k = 0$, so with $f := \prod_{i=1}^{r} f_i^k \neq 0$ we have $f(a_1, \ldots, a_n) = 0$, contradicting the algebraic independence of the a_i. Hence for some M_i the elements $a_1 + M_i, \ldots, a_n + M_i \in A/M_i$ are algebraically independent. It suffices to show that $\dim(A/M_i) \geq n$, so by replacing A by A/M_i, we may assume that A is an affine domain.

Consider the field $L := \operatorname{Quot}(K[a_1])$, which is a subfield of $\operatorname{Quot}(A)$, and the subalgebra $A' := L \cdot A \subseteq \operatorname{Quot}(A)$. Clearly A' is an affine L-domain, and $a_2, \ldots, a_n \in A'$ are algebraically independent over L. By induction, $\dim(A') \geq n - 1$, so there exists a chain

$$P'_0 \subsetneqq P'_1 \subsetneqq \cdots \subsetneqq P'_{n-1}$$

in $\operatorname{Spec}(A')$. Set $P_i := A \cap P'_i \in \operatorname{Spec}(A)$. Then $P_{i-1} \subseteq P_i$ for $i = 1, \ldots, n-1$. These inclusions are strict since clearly $L \cdot P_i = P'_i$ for all i. Moreover, $L \cap P_{n-1} = \{0\}$, since otherwise P'_{n-1} would contain an invertible element from L, leading to $P'_{n-1} = A'$. It follows that $a_1 + P_{n-1} \in A/P_{n-1}$ is not algebraic over K. By Lemma 1.1(b), A/P_{n-1} is not a field, so P_{n-1} is not a maximal ideal. Let $P_n \subset A$ be a maximal ideal containing P_{n-1}. Then we have a chain

$$P_0 \subsetneqq P_1 \subsetneqq \cdots \subsetneqq P_{n-1} \subsetneqq P_n$$

in $\operatorname{Spec}(A)$, and $\dim(A) \geq n$ follows. □

In Exercise 5.3, the scope of Theorem 5.9 will be extended to all subalgebras of affine algebras. In Chapter 8, we will learn more about chains of prime ideals in affine domains (see Theorem 8.22).

We will now use Theorem 5.9 in order to characterize 0-dimensional affine algebras. To avoid ambiguities, we write $\dim_K(V)$ for the dimension (= size of a basis) of a vector space V over a field K.

Theorem 5.11 (0-dimensional affine algebras)**.** *Let $A \neq \{0\}$ be an affine K-algebra. Then the following statements are equivalent:*

(a) $\dim(A) = 0$.
(b) A is algebraic over K.
(c) $\dim_K(A) < \infty$.
(d) A is Artinian.
(e) $|\operatorname{Spec}_{\max}(A)| < \infty$.

Proof. If $\dim(A) = 0$, then A is algebraic by Theorem 5.9. Assume that A is algebraic. We can write $A = K[a_1, \dots, a_n]$, so there exist nonzero polynomials $g_i \in K[x]$ with $g_i(a_i) = 0$. It is easy to see that the set

$$\left\{\prod_{i=1}^{n} a_i^{e_i} \mid 0 \leq e_i < \deg(g_i) \text{ for all } i\right\}$$

generates A as a K-vector space. Now assume that A is finite-dimensional as a K-vector space. Then the linear subspaces satisfy the descending chain condition. Therefore so do the ideals, so A is Artinian.

Assume that A is Artinian. By Corollary 3.14(a) and (b), every maximal ideal of A contains one of the minimal prime ideals $P_1, \dots, P_n$. But by Theorem 2.8, the P_i themselves are maximal. This implies (e).

Finally, assume that there exist only finitely many maximal ideals, and let $P \in \operatorname{Spec}(A)$. By Theorem 1.13, P is the intersection of all maximal ideals of A containing P, so P is a finite intersection of maximal ideals. Since P is a prime ideal, it follows that P itself is maximal. Therefore $\dim(A) = 0$. □

Exercise 5.4 gives an interpretation of $\dim_K(A)$ in the case that $A = K[X]$ is the coordinate ring of a finite set X. The following proposition describes 0-dimensional subsets of K^n. For K algebraically closed, this is just a reformulation of Theorem 5.11(e).

Proposition 5.12 (0-dimensional sets)**.** *Let K be a field and $X \subseteq K^n$ nonempty. Then* $\dim(X) = 0$ *if and only if X is finite.*

Proof. Assume $\dim(X) = 0$. Since X is a subset of a Noetherian space, X is Noetherian, too. By Theorem 3.11(a), X is a finite union of closed, irreducible subsets Z_i. Choose $x_i \in Z_i$. Then $\{x_i\} \subseteq Z_i$ is a chain of closed, irreducible subsets, so $Z_i = \{x_i\}$. It follows that X is finite.

Conversely, if X is finite, then the irreducible subsets are precisely the subsets of size 1, so $\dim(X) = 0$. □

The following theorem deals with a situation that is, in a sense, opposite to the one from Theorem 5.11: equidimensional algebras whose dimension is only 1 less than the number of generators. These correspond to equidimensional affine varieties in K^n of dimension $n - 1$. Such varieties are usually called *hypersurfaces*.

Theorem 5.13 (Hypersurfaces)**.** *Let $I \subseteq K[x_1, \ldots, x_n]$ be an ideal in a polynomial ring over a field, and $A := K[x_1, \ldots, x_n]/I$. Then the following statements are equivalent:*

(a) A is equidimensional of dimension $n - 1$.
(b) $I \neq K[x_1, \ldots, x_n]$, and every prime ideal in $K[x_1, \ldots, x_n]$ that is minimal over I is minimal among all nonzero prime ideals. (According to Definition 6.10, this means that I has height 1.)
(c) $\sqrt{I} = (g)$ with $g \in K[x_1, \ldots, x_n]$ a nonconstant polynomial.

Proof. In the proof we will make frequent use of the bijection between $\operatorname{Spec}(A)$ and $\mathcal{V}_{\operatorname{Spec}(K[x_1,\ldots,x_n])}(I)$ given by Lemma 1.22. Let $\mathcal{M} \subseteq \operatorname{Spec}(K[x_1, \ldots, x_n])$ be the set of all prime ideals that are minimal over I. Then $\mathcal{M}$ is finite by Corollary 3.14(d), and the minimal prime ideals of A are the P/I, $P \in \mathcal{M}$.

First assume that A is equidimensional of dimension $n-1$, so for all $P \in \mathcal{M}$ we have

$$\dim\left(K[x_1, \ldots, x_n]/P\right) = \dim\left(A/(P/I)\right) = n - 1. \tag{5.2}$$

It follows from Corollary 5.7 that $P \neq \{0\}$. If P were not minimal among all nonzero primes, we could build a chain of prime ideals in $\operatorname{Spec}(K[x_1, \ldots, x_n])$ by going two steps down from P, and, using (5.2), going $n-1$ steps up from P. This chain would have length $n + 1$, contradicting Corollary 5.7. Since (a) also implies that $I \neq K[x_1, \ldots, x_n]$, (b) follows.

Now assume (b), and again take $P \in \mathcal{M}$. By Lemma 5.14, which we prove below, there exists an irreducible polynomial g_P such that $P = (g_P)$. With Corollary 3.14(d), it follows that

$$\sqrt{I} = \bigcap_{P \in \mathcal{M}} P = \bigcap_{P \in \mathcal{M}} (g_P) = (g),$$

where we set $g := \prod_{P \in \mathcal{M}} g_P$. Since $I \neq K[x_1, \ldots, x_n]$, g is nonconstant, so (c) holds.

Finally, assume (c), and let $g = g_1 \cdots g_r$ be a decomposition into irreducible polynomials. For $i \neq j$, g_i does not divide g_j since (g) is a radical ideal. We obtain prime ideals $P_i := (g_i) \in \operatorname{Spec}(K[x_1, \ldots, x_n])$, and $\bigcap_{i=1}^r P_i = \sqrt{I}$. It follows that $\mathcal{M} = \{P_1, \ldots, P_r\}$. Since $\dim\left(A/(P_i/I)\right) = \dim\left(K[x_1, \ldots, x_n]/P_i\right)$ we need to show that $\dim\left(K[x_1, \ldots, x_n]/(g_i)\right) = n-1$ for all i. But it is clear that by excluding an indeterminate x_j that occurs in g_i from the set $\{x_1, \ldots, x_n\}$, we obtain a maximal subset of $\{x_1, \ldots, x_n\}$ that is algebraically independent modulo g_i. So the claim follows by Proposition 5.10 and Theorem 5.9. □

The following lemma was used in the proof. Recall that a *factorial ring* is the same as a *unique factorization domain*.

Lemma 5.14 (Height-one prime ideals in a factorial ring)**.** *Let R be a factorial ring and let $P \in \operatorname{Spec}(R)$ be prime ideal that is minimal among all* *nonzero prime ideals. (According to Definition 6.10, this means that P has height 1.) Then $P = (a)$ with $a \in R$ a prime element.*

Proof. Let $a \in P \setminus \{0\}$. Since P is a prime ideal, at least one factor of a factorization of a into prime elements also lies in P, so we may assume a to be a prime element. Then (a) is a prime ideal and $\{0\} \subsetneqq (a) \subseteq P$, so $(a) = P$ by the minimality hypothesis. □

Part (c) of Theorem 5.13 talks about principal ideals. This should be compared to Theorem 8.25, which talks about ideals generated by n polynomials. Readers may also take a look at Theorem 7.4, where the implication (c) ⇒ (b) of Theorem 5.13 is generalized from $K[x_1, \ldots, x_n]$ to arbitrary Noetherian rings.

As a further application of Theorem 5.9 and Proposition 5.10, we determine the dimension of a product of affine varieties.

Theorem 5.15 (Dimension of a product variety)**.** *Let $X \subseteq K^n$ and $Y \subseteq K^m$ be nonempty affine varieties over an algebraically closed field K. Then the product variety $X \times Y \subseteq K^{n+m}$ satisfies*

$$\dim(X \times Y) = \dim(X) + \dim(Y).$$

Proof. The proof is very easy and straightforward, even if it takes some space to write it down.

Write $d = \dim(X) = \dim(K[X])$ and $e = \dim(Y) = \dim(K[Y])$. By Theorem 5.9 and Proposition 5.10, d is the largest nonnegative integer m such that there exist pairwise distinct indeterminates $x_{i_1}, \ldots, x_{i_m} \in \{x_1, \ldots, x_n\}$ such that

$$\{f \in K[x_{i_1}, \ldots, x_{i_m}] \mid f \in \mathcal{I}(X)\} = \{0\}. \tag{5.3}$$

So we have $x_{i_1}, \ldots, x_{i_d} \in \{x_1, \ldots, x_n\}$ and $y_{j_1}, \ldots, y_{j_e} \in \{y_1, \ldots, y_m\}$ (with $y_1, \ldots, y_m$ a new set of indeterminates) satisfying (5.3) for X and Y, respectively. To show that the union of these satisfy (5.3) for $X \times Y$, let $f \in K[x_{i_1}, \ldots, x_{i_d}, y_{j_1}, \ldots, y_{j_e}]$ be a polynomial that vanishes on $X \times Y$. Write $f = \sum_{k=1}^r g_k t_k$ with $g_k \in K[x_{i_1}, \ldots, x_{i_d}]$ and t_k pairwise distinct products of powers of the y_{j_ν}. Let $(\xi_1, \ldots, \xi_n) \in X$. Then the polynomial $\sum_{k=1}^r g_k(\xi_1, \ldots, \xi_n) t_k \in K[y_{j_1}, \ldots, y_{j_e}]$ lies in $\mathcal{I}(Y)$, so it is zero. Since the t_k are linearly independent over K, this implies $g_k(\xi_1, \ldots, \xi_n) = 0$ for all k. Since this holds for all points in X, we conclude $g_k = 0$, so $f = 0$. This shows that $\dim(X \times Y) \geq d + e$.

To see that $\dim(X \times Y)$ is not greater than $d + e$, let $T \subseteq \{x_1, \ldots, x_n, y_1, \ldots, y_m\}$ be a subset with $|T| > d + e$. Then $|T \cap \{x_1, \ldots, x_n\}| > d$ or $|T \cap \{y_1, \ldots, y_m\}| > e$. By symmetry, we may assume the first case, so there exist pairwise distinct $x_{i_1}, \ldots, x_{i_m} \in T$ with $m > e$. Therefore we have $f \in K[x_{i_1}, \ldots, x_{i_m}] \setminus \{0\}$ which vanishes on X. So f, viewed as a polynomial in the indeterminates from T, vanishes on $X \times Y$. This completes the proof. □

Exercises for Chapter 5

5.1 (The dimension of a subset). Let X be a topological space and let $Y \subseteq X$ be a subset equipped with the subset topology.

(a) Show that Y is irreducible if and only if the closure $\overline{Y}$ is irreducible.
(b) Show that $\dim(Y) \leq \dim(X)$.

5.2 (Dimension of the power series ring). Let K be a field and $R = K[[x]]$ the formal power series ring over K. Show that $\dim(R) = 1$.

***5.3 (Subalgebras of affine algebras).** Let A be a (not necessarily finitely generated) subalgebra of an affine algebra. Show that Theorem 5.9 and Proposition 5.10 hold for A. *(Solution on page 218)*

5.4 (Coordinate rings of finite sets of points). Let K be a field and $X \subseteq K^n$ a finite set of points. Show that

$$\dim_K (K[X]) = |X|.$$

5.5 (The ring of Laurent polynomials). Let K be a field, $K(x)$ the rational function field, and $R = K[x, x^{-1}] \subset K(x)$ the ring of Laurent polynomials. Determine the Krull dimension of R.

5.6 (Right or wrong?). Decide whether each of the following statements is true or false. Give reasons for your answers.

(a) If $R \subseteq S$ is a subring, then $\dim(R) \leq \dim(S)$.
(b) If A is an affine algebra and $B \subseteq A$ a subalgebra, then $\dim(B) \leq \dim(A)$.
(c) If R is a ring and $I \subseteq R$ an ideal, then $\dim(R/I) \leq \dim(R)$.
(d) If A is an affine K-algebra, then the transcendence degree of A is the size of a maximal algebraically independent subset of A.
(e) If A is an affine K-domain, then the transcendence degree of A is the size of a maximal algebraically independent subset of A.
(f) Let A be a zero-dimensional algebra over a field K. Then $\dim_K(A) < \infty$.

5.7 (Matrices of small rank). Let K be an infinite field and $K^{n\times m}$ the set of all $n \times m$ matrices with entries in K, which we identify with affine $n \cdot m$-space $K^{n\cdot m}$. For an integer k with $0 \leq k \leq \min\{n, m\}$, let

$$X_k := \left\{A \in K^{n\times m} \mid \operatorname{rank}(A) \leq k\right) \subseteq K^{n\times m}.$$

(a) Show that X_k is closed and irreducible.
Hint: Pick a matrix $M \in K^{n\times m}$ of rank k and consider the map $f\colon K^{n\times n} \times K^{m\times m} \to K^{n\times m},\ (A, B) \mapsto AMB$.

(b) Show that

$$\dim(X_k) = k \cdot (n + m - k).$$

Hint: Determine the transcendence degree of $K[X_k]$ using (a), Exercise 5.6(e), and Remark 5.4(a).

5.8 (Images of morphisms). Let X and Y be affine varieties over an algebraically closed field K, $f\colon X \to Y$ a morphism, and $\overline{\operatorname{im}(f)}$ the Zariski closure of its image. Show that

$$\dim\left(\overline{\operatorname{im}(f)}\right) \le \dim(X). \tag{5.4}$$

Does (5.4) extend to the case that f is a morphism of spectra?
Remark: By Exercise 5.1, the inequality (5.4) implies $\dim(\operatorname{im}(f)) \le \dim(X)$.

5.9 (The polynomial ring over a principal ideal domain). Let R be a principal ideal domain that is not a field. Show that the polynomial ring $R[x]$ has dimension 2.
Hint: For a chain of prime ideals P_i in $R[x]$, consider the ideals in $\operatorname{Quot}(R)[x]$ generated by the P_i. Show that $R \cap P_2 \neq \{0\}$.
Remark: This result is a special case of Corollary 7.13 on page 84, which requires much more work.