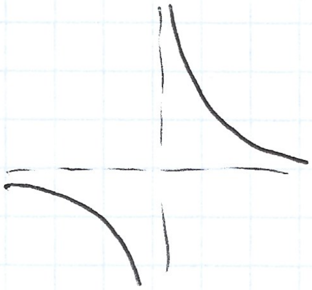
Lecture 10

Recall:

$$\left\{\begin{matrix}\text{morphisms}\\ X \xrightarrow{\varphi} Y\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{algebra homomorphisms}\\ A(Y) \xrightarrow{\varphi^*} A(X)\end{matrix}\right\}$$

E.g.



$X = \{xy = 1\} \subset \mathbb{A}^2$

$Y = \mathbb{A}^1_x$ $\quad \varphi : X \to Y$

$\qquad (x,y) \to x$

Note $\varphi(X) = \{x \neq 0\}$ not closed in Y

But $A(X) = \dfrac{\mathbb{k}[x,y]}{(xy-1)} \simeq \mathbb{k}[x, x^{-1}] \xleftarrow{\varphi^*} A(Y) = \mathbb{k}[x]$

$\varphi^*(x) = x$

φ^* is injective.

Def A morphism $\varphi : X \to Y$ is called dominant if the image of X is dense in Y, that is, $\overline{\varphi(X)} = Y$.

Thm $\varphi : X \to Y$ is dominant $\iff$ $\varphi^* : A(Y) \to A(X)$ is injective.

Proof: ① Assume φ^* is injective, need to prove $\overline{\varphi(X)} = Y$.

Suppose $g = 0$ on $\varphi(X)$ then $\varphi(X) \subset \{g = 0\}$.

On the other hand, $g(\varphi(x)) = 0$ $\forall x \in X \Rightarrow \varphi^* g = 0$

But φ^* is injective $\Rightarrow g = 0$ in $A(Y) \Rightarrow Y \subset \{g = 0\}$.

Therefore $\overline{\varphi(X)} = Y$.

② Assume φ^* is not injective and $\varphi^* g = 0$ in $A(X)$.

but $g \neq 0$ in $A(Y)$. This means $g(\varphi(x)) = 0$ $\forall x \in X$

$\Rightarrow \varphi(X) \subset \{g = 0\} \Rightarrow \overline{\varphi(X)} \subset \{g = 0\}$. But $Y \not\subset \{g = 0\}$

So $\overline{\varphi(X)} \neq Y$ and φ is not dominant. $\square$

Thm φ^* surjective $\iff$ $\varphi(X)$ closed and φ is an isomorphism onto its image (closed embedding)

Proof ① Assume φ^* surjective, $Z = \overline{\varphi(X)}$ closed in Y

We get a diagram of maps:

$$\begin{tikzcd} X \arrow[d, "\alpha"'] \arrow[dr, "\varphi"] & \\ Z \arrow[r, hook] & Y \end{tikzcd} \qquad \begin{tikzcd} A(X) & \\ A(Z) \arrow[u, hook, "\alpha^*"] & A(Y) \arrow[l, two heads, "\text{surjective}"] \arrow[ul, two heads, "\varphi^*"', "\text{surjective}"] \end{tikzcd}$$

injective since $\alpha : X \to Z$ dominant

$A(Z) = \dfrac{A(Y)}{J}$

$A(Y) \twoheadrightarrow A(Z)$ surjective.

Therefore $\alpha^* : A(Z) \to A(X)$ is surjective and injective

$\Rightarrow$ α is an isomorphism and $Z = \overline{\varphi(X)} = \varphi(X)$.

② Assume $\underset{\cong Z}{\varphi(X)}$ is closed and $X \simeq Z$. Then we get

$$\begin{tikzcd} X \arrow[d, "\wr"'] \arrow[dr, "\varphi"] & \\ Z \arrow[r] & Y \end{tikzcd} \qquad \begin{tikzcd} A(X) & \\ A(Z) \arrow[u, "\wr"] & A(Y) \arrow[l, two heads] \arrow[ul, "\varphi^*"'] \end{tikzcd}$$

Therefore φ^* is surjective
