## Lecture 11

We need to talk about rings of functions for more general sets.

**Def** Suppose $g \in A(X)$. We define the principal open subset $D(g) = \{g \neq 0\}$.

**Def** The ring of functions on $D(g)$ is defined as

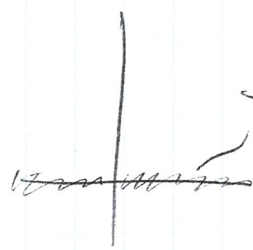
$$A(D(g)) = A(X)[g^{-1}] = \left\{ \frac{f}{g^k} ; k \geq 0, f \in A(X) \right\} \leftarrow \text{localization}$$

This is a ring: $\frac{f_1}{g^k} + \frac{f_2}{g^m} = \frac{f_1 g^m + f_2 g^k}{g^{k+m}}$, $\frac{f_1}{g^k} \cdot \frac{f_2}{g^m} = \frac{f_1 f_2}{g^{k+m}}$.

Also $\frac{f_1}{g^k} \sim \frac{f_2}{g^m}$ iff $f_1 g^{m-k} - f_2 = 0$ in $A(X)$.
$k \leq m$

**Warning**: If $g$ is a zero divisor then $A(D(g))$ does not contain $A(X)$.

**Ex** $X = \{xy = 0\}$



$D(x) = \{x \neq 0\} \cap \{xy = 0\}$

$$y \sim \frac{xy}{x} = \frac{0}{x} = 0$$

Also, if $xy = 0$ and $x \neq 0$ then $y = 0$.

So the map $\frac{\mathbb{K}[x,y]}{(xy)} \longrightarrow \frac{\mathbb{K}[x,y,x^{-1}]}{(xy)} \simeq \mathbb{K}[x,x^{-1}]$ is not injective.

Still, we have a homomorphism $A(X) \longrightarrow A(D(g))$

corresponding to the map $D(g) \hookrightarrow X$.

Thus Consider $Z \subset X \times \mathbb{A}^1_t$ defined by the equation $\{g(x) \cdot t = 1\}$. Then $Z \simeq D(g)$.

In other words, $D(g)$ = open in $X$ ⟹ isomorphic to a closed subset of $X \times \mathbb{A}^1$.

Proof ① Geometrically:

$$D(g) \xrightarrow{\varphi_1} Z \qquad\qquad Z \xrightarrow{\varphi_2} D(g)$$

$$x \longrightarrow \left(x, \underset{\substack{\shortparallel \\ t}}{\frac{1}{g(x)}}\right) \qquad\qquad (x,t) \longrightarrow x$$

regular function on $D(g)$

Since $t = \frac{1}{g}$, $\varphi_1 \circ \varphi_2 = Id$ and $\varphi_2 \circ \varphi_1 = Id$ so these are inverse to each other.

② Algebraically:

$$A(Z) = \frac{A(X)[t]}{(g(x)t - 1)} \underset{\varphi_2^*}{\overset{\varphi_1^*}{\rightleftarrows}} A(D(f)) = \left\{\frac{f}{g^k}\right\}$$

$$f \cdot t^k \longleftrightarrow \frac{f}{g^k}$$

Claim $\varphi_1^*$ and $\varphi_2^*$ are well defined

Pf. • $\varphi_1^*(gt - 1) = \frac{g}{g} - 1 = 0$ ✓

• Assume $\frac{f_1}{g^k} \sim \frac{f_2}{g^m}$, $k \leq m$, then $f_1 g^{m-k} - f_2 = 0$ in $A(X)$

On the other hand, ~~for~~

$$\varphi_2^*\left(\frac{f_1}{g^k} - \frac{f_2}{g^m}\right) = f_1 t^k - f_2 g^m =$$

$$= t^k\left(f_1 - f_2 t^{m-k}\right) = t^k\left(f_1 - f_1 g^{m-k} t^{m-k}\right)$$

$$= f_1 t^k\left(1 - g^{m-k} t^{m-k}\right).$$

Since $1-g^{m-k}t^{m-k}$ is divisible by $1-gt$

$\varphi_2^*\left(\frac{f_1}{g^k} - \frac{f_2}{g^m}\right) \in (1-gt)$ so vanishes on $A(Z)$.

□

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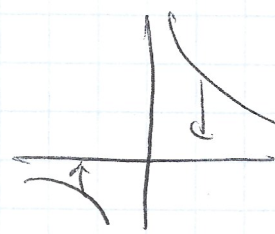
**Ex** $\underset{\text{open}}{\{x \neq 0\} \subset \mathbb{A}^1} \simeq \underset{\text{closed}}{\{xt = 1\} \subset \mathbb{A}^2}$



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**Def** $W$ is called *quasi-affine* if it is an open subset in an affine algebraic set.

**Lemma** If $X$ = alg. set, $W \subset X$ open then

$W = \bigcup_{i=1}^{r} D(g_i)$ finite union of principal open subsets

**Pf** $X - W = Z$ closed.

Since $A(X)$ is Noetherian, $I(Z)$ is finitely generated by $g_1 \dots g_r$. then $Z = \{g_1 = \dots = g_r = 0\} \subset X$

$$= \{g_1 = 0\} \cap \dots \cap \{g_r = 0\}$$

$$W = X - Z = \{g_1 \neq 0\} \cup \dots \cup \{g_r \neq 0\}.$$

□

Q How to describe regular function on $W$?

Def 1 $W = \bigcup_i D(g_i) \underset{\text{open}}{\subset} X$, $X$ = alg. set.

A function $\varphi: W \longrightarrow \mathbb{K}$ is regular if $\varphi|_{D(g_i)}$ is regular for all $i$.

Ex $W = \mathbb{A}^2 \setminus \{(0,0)\} = D(x) \cup D(y)$

either $x \neq 0$ or $y \neq 0$

$\varphi: W \longrightarrow \mathbb{K}$ regular if $\varphi|_{D(x)} = \frac{f_1}{x^k}$ $\quad \varphi|_{D(y)} = \frac{f_2}{y^m}$

On $D(x) \cap D(y)$ we have

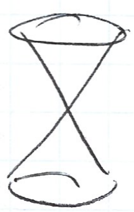
$$\frac{f_1}{x^k} = \frac{f_2}{y^m} \Rightarrow f_1 y^m = f_2 x^k$$

Since $GCD(x,y) = 1$ and $\mathbb{K}[x,y]$ is UFD, we conclude that $f_1$ is divisible by $x^k$ and $f_2$ by $y^m$, so

$\varphi = \frac{f_1}{x^k}$ is a polynomial.

Conclusion Any regular function on $\mathbb{A}^2 \setminus \{(0,0)\}$ is a polynomial $\Rightarrow$ extends to a regular function on $\mathbb{A}^2$.

Ex $X = \{xy = z^2\} \subset \mathbb{A}^3$ $\quad \frac{x}{z} = \frac{z}{y}$



$\Rightarrow \varphi = \frac{x}{z}$ is regular on $W = \{y \neq 0\} \cup \{z \neq 0\}$

$= X - \{y = z = 0\}$.