Lecture 12 Recap: W is quasi affine if

W is an open subset of some algebraic set X.
$W \underset{\text{open}}{\subset} X \underset{\text{cl.}}{\subseteq} \mathbb{A}^n$
We want to define regular function on W.

Def 1 $W = \bigcup_{i=1}^{r} D(g_i)$, $\varphi : W \longrightarrow \mathbb{K}$ is regular if $\varphi|_{D(g_i)}$ is regular for all i.

Def 2 • φ is regular at a point $p \in W$ if there exists an open $p \in U \subset W$ and a rational funct. $\frac{g}{h}$, $g, h \in \mathbb{K}[x_1 \dots x_n]$ such that ① $h \neq 0$ on U
② $\varphi = \frac{g}{h}$ on U

• φ is regular on W if it is regular at every point of W.
$\mathcal{O}(W)$ = ring of regular functions with Def 2

Thm (a) Suppose W = alg. set, then $\mathcal{O}(W) \simeq A(W)$
(b) Suppose $W = D(g) \subset X$, then $\mathcal{O}(W) = A(X)[g^{-1}]$
(c) Two definitions of $\mathcal{O}(W)$ agree.

Proof: (a) Clearly, $A(W) \subset \mathcal{O}(W)$. Suppose $\varphi \in \mathcal{O}(W)$
Define $J = \{h \in \mathbb{K}[x_1 \dots x_n] : h\varphi \in A(W)\} \supset I(W)$

For all $p \in W$ $\exists h : h(p) \neq 0$, $h\varphi \in A(W) \Rightarrow h \in J$

Therefore $Z(J) = \emptyset \Rightarrow$ by Nullstellensatz

$J = \mathbb{k}[x_1 \dots x_n] \Rightarrow 1 \in J$ and $1 \cdot \varphi = \varphi \in A(W)$.

(b) Follows from (a) and $D(g) \cong \{gt = 1\} \subset X \times \mathbb{A}^1$.

(c) Assume $W = \bigcup_i D(g_i) \subset X$

- If φ is regular by $\boxed{\text{Def 1}}$ then $\varphi|_{D(g_i)}$ is regular for all i and $\varphi|_{D(g_i)} = \frac{f_i}{g_i^k}$, $f_i \in \mathbb{k}[x_1 \dots x_n]$

therefore φ is regular with $\boxed{\text{Def 2}}$ at all points of $D(g_i)$

$\Rightarrow$ regular by $\boxed{\text{Def 2}}$ at all points of W.

- Suppose φ is regular by $\boxed{\text{Def 2}}$ at all point of W

Given $p \in D(g_i)$, $\exists U \subset W$ open such that $\varphi|_U = \frac{f}{h}$, $h \neq 0$ on U. Since $U \cap D(g_i)$ is open in $D(g_i)$, we get that φ is regular on $D(g_i)$ by $\boxed{\text{Def 2}}$

Now by (b) φ is regular by $\boxed{\text{Def 1}}$ on $D(g_i)$ and the result follows. $\square$

Def $W \subset \mathbb{P}^n$ is quasi-projective if $W \underset{\text{open}}{\subset} X$, X = projective alg. set.

Def Assume W is quasiprojective. $\varphi : W \to \mathbb{k}$ is regular if at all point p there is an open

$p \in U \subset W$ such that $\varphi = \frac{f}{g}$ on U, $g \neq 0$ on U
and f, g are homogeneous polynomials of the same degree

Note
$$\frac{f(\lambda x_0, \dots \lambda x_n)}{g(\lambda x_0 \dots \lambda x_n)} = \frac{\lambda^d f(x_0 \dots x_n)}{\lambda^d g(x_0 \dots x_n)} = \frac{f(x_0 \dots x_n)}{g(x_0 \dots x_n)}$$

Def $V \subset \mathbb{P}^n$, $W \subset \mathbb{P}^m$ quasi-projective, $\Phi : V \to W$ regular if for all p $\exists$ open $p \in U \subset V$ such that

$$\Phi(x_0 : \dots : x_n) = [\varphi_0(x), \dots \varphi_m(x)]$$

and φ_i are homogeneous polynomials of the same degree

and φ_i do not vanish simultaneously on U.

Note ① If $\varphi_i(x) \neq 0$ then

$$[\varphi_0(x) : \dots : \varphi_n(x)] = \left[\frac{\varphi_0}{\varphi_i} : \dots : 1 : \dots \frac{\varphi_m}{\varphi_i}\right]$$

a map $\Phi : U \to \mathbb{A}^m = \{\varphi_i \neq 0\}$

components = regular functions

② If Φ is given by rational functions

$$\Phi = \left[\frac{f_0}{g_0} : \dots : \frac{f_m}{g_m}\right]$$ $\deg f_i = \deg g_i = d_i$
clear denominators

Then $\Phi = [f_0 g_1 \dots g_m : \dots : f_m g_0 \dots g_{m-1}]$

polynomials of same degree $d_0 + \dots + d_m$

__Ex__: $C = \{xy = z^2\} \subset \mathbb{P}^2$ w. coords $[x:y:z]$

$\mathbb{P}^1$ with coord $[a:b]$

① Define map $\mathbb{P}^1 \xrightarrow{\Phi} C$

$[a:b] \longrightarrow [a^2 : b^2 : ab] \leftarrow$ all degree 2

$a^2 \cdot b^2 = (ab)^2 \Rightarrow \Phi(\mathbb{P}^1) \subset C$.

② Define $C \xrightarrow{\Psi} \mathbb{P}^1$

$$[x:y:z] \longrightarrow \begin{cases} [x:z] & \text{if } x \neq 0 \\ [z:y] & \text{if } y \neq 0 \end{cases}$$

If $x=y=0$ then $z=0$ (from $xy=z^2$) $\Rightarrow [x:y:z]$ undefined.

So $C = \underbrace{[C \cap \{x \neq 0\}] \cup [C \cap \{y \neq 0\}]}_{\text{open in } C}$

On the intersection we get $\underset{\substack{x\neq 0,\, y\neq 0 \\ z \neq 0}}{}$ $[x:z] \sim [xy:yz] \overset{\text{from the eq.}}{\sim} [z^2 : yz] \sim [z:y]$.

So Ψ is a well defined morphism to $\mathbb{P}^1$.

__Exercise__ Φ and Ψ are inverse to each other, so $C \simeq \mathbb{P}^1$.

In fact (HW 4) any nondegenerate quadric in $\mathbb{P}^2$ is isomorphic to $\mathbb{P}^1$.