Lecture 13 | Two important constructions

1 Quadrics in $\mathbb{P}^2$

$\{F(x_0, x_1, x_2) = 0\} = X \subset \mathbb{P}^2$ (F homogeneous degree 2)

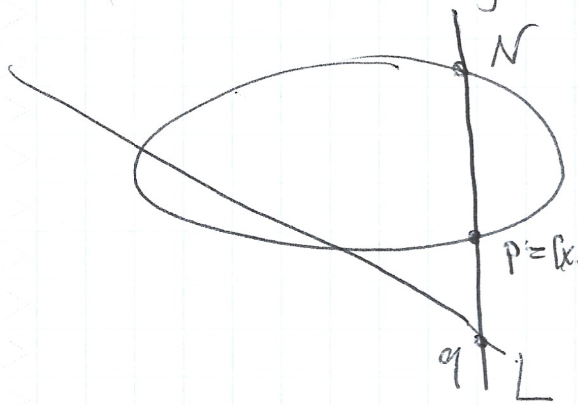
$\mathbb{K}$ = any field (ex. $\mathbb{Q}$), char $\neq 2$

Assume (a) $X \neq \emptyset$ (b) F is irreducible

Lemma F does not contain lines

Pf Assume $\{\ell = 0\} \subset \{F = 0\}$, then $F \in I(\ell)$

$\Rightarrow F$ is divisible by ℓ, but F is irreducible, contradiction.



Construction: pick a point $N \in X$

Pick a line L not containing N

Given $p \in X$, we define $q \in L$ as the point of intersection of the line Np with L.

Note: $N \notin L \Rightarrow Np \neq L \Rightarrow$ lines Np and L intersect exactly at one point in $\mathbb{P}^2$.

Claim: If $p = [x_0 : x_1 : x_2]$ then the coordinates of q are rational functions of x_0, x_1, x_2.

Proof: $q = \lambda p + \mu N$, if we plug it into the linear equation of L we get a linear equation

$c_1 \lambda + c_2 \mu = 0$ where c_1, c_2 = homogeneous polynomials in x_0, x_1, x_2

Solution: $[\lambda : \mu] = [-c_2 : c_1]$

Conversely, given $q \in L$, we define $p = Nq \cap X$

The line Nq intersects X in at most 2 points since it is not contained in X by Lemma; one such point is N so we define p as the other intersection point.

Claim If $q = [y_0 : y_1 : y_2]$ then the coordinates of p are rational function of y_i.

Proof Similarly, to find p we need to solve a quadratic equation $a(y)\left(\frac{\lambda'}{\mu'}\right)^2 + b(y)\left(\frac{\lambda'}{\mu'}\right) + c(y) = 0$ | $p = \lambda' q + \mu' N$

where $a(y) \neq 0$ and we know one root $[\lambda' : \mu'] = [0 : 1] \Rightarrow c(y) = 0$

And we know the other root $-\frac{b(y)}{a(y)}$ given by rational fn.

Subtlety We can have a double root, when the line is tangent to X

but this is ok, in this case $p = N$.

Conclusion We constructed an explicit isomorphism $X \cong L$

So any irreducible quadric in $\mathbb{P}^2(\mathbb{K})$ is isomorphic to $\mathbb{P}^1(\mathbb{K})$

$\boxed{2}$ Blow-up $Z \subset \mathbb{A}^2_p \times \mathbb{P}^1_\ell$ consists of pairs (p, ℓ) such that (a) $p \in \mathbb{A}^2$ point

(b) $\ell \in \mathbb{P}^1$ line

(c) $p \in \ell$.

Claim: $Z \subset \mathbb{A}^2 \times \mathbb{P}^1$ is a closed subset

Proof: $p = (x,y)$ $\ell = [u:v]$

$p \in \ell$ iff $\det\begin{bmatrix} x & y \\ u & v \end{bmatrix} = 0$ (x,y) is collinear to (u,v)

$xv - uy = 0$.   ← polynomial in x, y homogeneous in u, v

$$\begin{tikzcd}
& Z \subset \mathbb{A}^2 \times \mathbb{P}^1 \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\
\mathbb{A}^2 & & \mathbb{P}^1
\end{tikzcd}$$

① $\beta : Z \to \mathbb{P}^1$ $(p, \ell) \to \ell$ $(x, y, [u:v]) \to [u:v]$

$\beta^{-1}(\ell) = \{$all points (x,y) in $\ell\} \simeq \mathbb{A}^1$ for all ℓ

In coordinates: On $\{u \neq 0\}$ $xv - uy = 0$

$y = \frac{xv}{u}$ $\frac{v}{u}$ = coord on $\mathbb{P}^1$

$\beta^{-1}(\{u \neq 0\}) \simeq \{u \neq 0\} \times \mathbb{A}^1$

$(x, \frac{xv}{u}, [u:v]) \leftarrow [u:v], x$

$(x, y, [u:v]) \to [u:v], x$

Similarly, $\beta^{-1}(\{v \neq 0\}) = \{v \neq 0\} \times \mathbb{A}^1$.

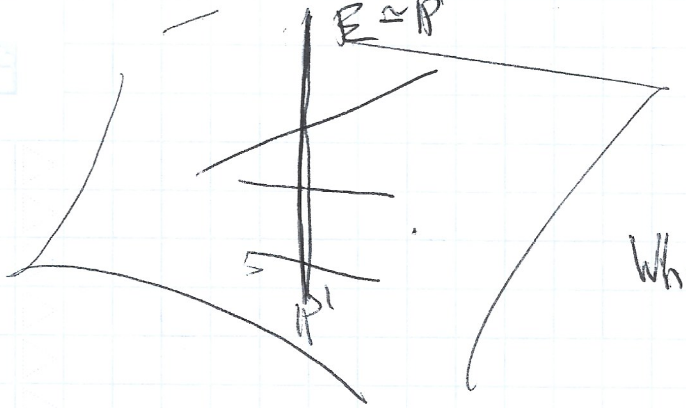
locally trivial fibration in Zariski topology (more next time)

② $\alpha : Z \to \mathbb{A}^2$ $(p, \ell) \to p$

$(x,y) \neq 0 \Rightarrow \exists$ unique line $[u:v] = [x:y]$ $\alpha^{-1}(x,y) = \{pt\}$

$(x,y) = 0 \Rightarrow$ any line $\alpha^{-1}(0,0) = \mathbb{P}^1$

Exceptional divisor

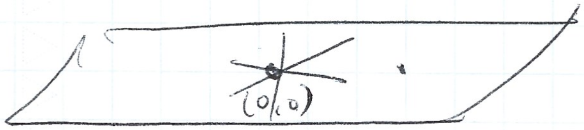


Def Z = blowup of $\mathbb{A}^2$ at $(0,0)$.

What is $\alpha^{-1}(\text{line in } \mathbb{A}^2) = \alpha^{-1}(y=kx)$

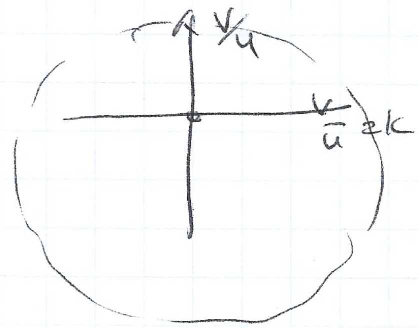
$$\begin{cases} xv = yu \\ y = kx \end{cases}$$

Let's work in charts.



$\{u \neq 0\}$ $\quad y = \frac{xv}{u} = kx$

$$x\left(\frac{v}{u} - k\right) = 0$$



$\{v \neq 0\}$ $\quad x = \frac{yu}{v}$, $y = kx$

$$y = \frac{kuy}{v} \qquad y\left(1 - \frac{ku}{v}\right) = 0$$

$y = 0$ or $\frac{u}{v} = \frac{1}{k} \sim \frac{v}{u} = k$.

$$\alpha^{-1}(\text{line}) = E \cup \{\text{line } [u:v] = [1:k]\} \subset Z$$

HW What about α^{-1}(other curves in $\mathbb{A}^2$)?

Can we define maps back?

$$\begin{tikzcd} & Z \arrow[dl, "\alpha"] \arrow[dr, "\beta"] & \\ \mathbb{A}^2 \arrow[ur, dashed, bend left, "\bar{\alpha}"] & & \mathbb{P}^1 \arrow[ul, bend right, "\bar{\beta}"'] \end{tikzcd}$$

$\bar{\beta}[u:v] = (0,0,[u:v])$

"zero section"

$\bar{\beta}(\mathbb{P}^1) = E$.

$$\bar{\alpha}: (x,y) \mapsto \begin{cases} (x,y,[1:\frac{y}{x}]) & \text{if } x \neq 0 \\ (x,y,[\frac{x}{y}:1]) & \text{if } y \neq 0 \\ \underline{\text{undefined}} & \text{if } x = y = 0 \end{cases}$$

So $\bar{\alpha}$ defined on open subset $\mathbb{A}^2 \setminus \{(0,0)\}$.