Lecture 14

Def $\varphi : V \dashrightarrow W$ is a rational map if it is a regular map $U \xrightarrow{\varphi} W$ where U is open and dense in V. (U = domain of φ)

Equivalently, φ is locally given by rational functions

$U = \{$all denominators $\neq 0\}$

Ex Z = Blowup of $\mathbb{A}^2$ at $(0,0) \subset \mathbb{A}^2 \times \mathbb{P}^1$

$= \{(x, y, [u:v]) : xv = uy\}$

Define $\varphi : \mathbb{A}^2 \setminus \{(0,0)\} \longrightarrow Z$

$$\varphi(x,y) = \begin{cases} (x, y, [1 : \frac{y}{x}]) & \text{if } x \neq 0 \\ (x, y, [\frac{x}{y} : 1]) & \text{if } y \neq 0 \\ \text{undefined at } (0,0). \end{cases}$$

φ is regular on $\mathbb{A}^2 \setminus \{(0,0)\}$ and rational $\mathbb{A}^2 \dashrightarrow Z$.

Pedantic remark: Assume V is irreducible. A rational map is an equivalence class of pairs $(U, \varphi : U \to W)$

$(U, \varphi) \sim (U', \varphi')$ if $\varphi = \varphi'$ on $U \cap U'$.

Def V and W are birational if

$$\begin{tikzcd} V \arrow[r, dashed, bend left, "\varphi"] & W \arrow[l, dashed, bend left, "\psi"] \end{tikzcd}$$

such that $\varphi \circ \psi = \psi \circ \varphi = \mathrm{Id}$ wherever defined.

Equivalently: open subset in V is isomorphic to an open subset in W.

$\underline{\underline{\text{Ex}}}$ $\mathbb{A}^2$ is birational to $Z = Bl_0 \mathbb{A}^2$
but $\mathbb{A}^2 \not\cong Z$.

<u>More generally:</u> $Bl_0 \mathbb{A}^n = \{p \in \mathbb{A}^n, \ell \in \mathbb{P}^{n-1} : p \in \ell\} \subset \mathbb{A}^n \times \mathbb{P}^{n-1}$

blowup of $\mathbb{A}^n$ at $(0,0)$

Coordinates $(x_1 \ldots x_n, [u_1 : \ldots : u_n])$

Equations: $\operatorname{rk}\begin{pmatrix} x_1 & \cdots & x_n \\ u_1 & \cdots & u_n \end{pmatrix} = 1 \Rightarrow \begin{vmatrix} x_i & x_j \\ u_i & u_j \end{vmatrix} = 0 \quad \forall i,j$

$\binom{n}{2}$ equations in $\mathbb{A}^n \times \mathbb{P}^{n-1}$

$$\begin{tikzcd} & Bl_0 \mathbb{A}^n \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ \mathbb{A}^n & & \mathbb{P}^{n-1} \end{tikzcd}$$

(1) $\beta^{-1}(\ell) \cong \mathbb{A}^1$, and $\beta^{-1}(\{u_i \neq 0\}) \cong \{u_i \neq 0\} \times \mathbb{A}^1$

indeed, if $u_i \neq 0$ then $x_j = x_i \cdot \frac{u_j}{u_i}$

so $\beta^{-1}(\{u_i \neq 0\})$ is affine space with coordinates

x_i and $\frac{u_j}{u_i}$, $j \neq i$.

(2) $\alpha^{-1}(p) = \begin{cases} \text{one point}, & p \neq (0 \ldots 0) \\ \mathbb{P}^{n-1} = E, & p = (0 \ldots 0) \end{cases}$

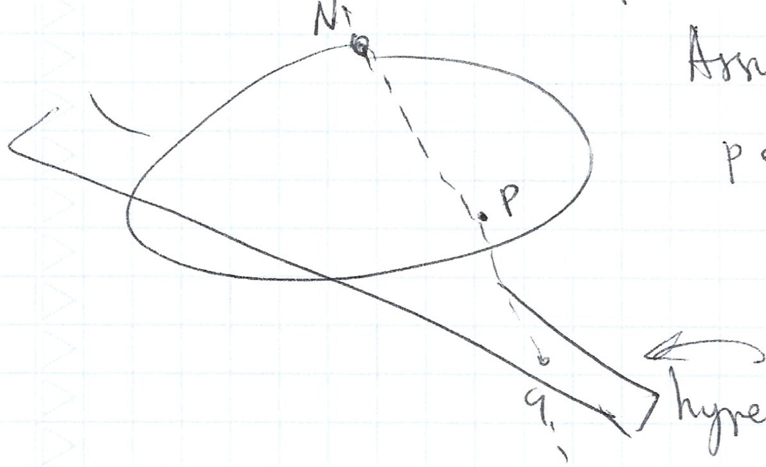
$(Bl_0 \mathbb{A}^n \setminus E) \cong (\mathbb{A}^n \setminus (0 \ldots 0))$ (isomorphic)

So $Bl_0 \mathbb{A}^n$ and $\mathbb{A}^n$ are <u>birational</u>.

Example Quadrics in $\mathbb{P}^n$

$X = \{F(x_0, \dots x_n) = 0\}$

irreducible quadratic polynomial



Assume $X \neq \emptyset$, $N \in X$

$p \in X$, $q \in L$

hyperplane $L = \{a_0 x_0 + \dots + a_n x_n = 0\}$

not containing X.

Given $p \in X$, find $q = Np \cap L \Rightarrow$ defined if $p \neq N$

Given $q \in L$, find $p = Nq \cap L$. $\Rightarrow$ defined if

(a) Nq is not contained in X

(b) Nq is not tangent to X

Lemma The set of q satisfying (a) and (b) is Zariski open in L.

Proof (sketch) $q = [y_0 : \dots : y_n]$, parametrize the line Nq:

$p = \lambda [y_0 \dots y_n] + \mu N$, need

$0 = F(\lambda [y_0 \dots y_n] + \mu N) = a(y)\lambda^2 + b(y)\lambda\mu + c(y)\mu^2$

$[\lambda : \mu] = [0 : 1]$ is a solution $\Rightarrow c(y) \equiv 0$

* Nq is contained in $X \Leftrightarrow a(y) = b(y) = 0$

* Nq is tangent to $X \Rightarrow$ we have a double root $\Rightarrow b(y) = 0$

So (a) + (b) $\Leftrightarrow b(y) \neq 0$ open in L.

Conclusion X and L are birational. (but not isomorphic for $n > 2$.)

Fun facts: What if $\deg F = 3$?

* ~~cubic curve~~ (Smooth) cubic curve in $\mathbb{P}^2$ is not rational ($\Leftrightarrow$ not birational to $\mathbb{P}^1$). Proof: later in class!
* (Smooth) cubic surface in $\mathbb{P}^3$ is rational ($\Leftrightarrow$ birational to $\mathbb{P}^2$)
* (Smooth) cubic 3-fold in $\mathbb{P}^4$ is not rational
* Cubic hypersurface in $\mathbb{P}^n$, $n > 4$ $\Rightarrow$ not known!

Lemma Any regular function on $\mathbb{P}^n$ is a constant.

Pf Assume $\varphi : \mathbb{P}^n \to K$ is regular

Define $\tilde{\varphi} : \mathbb{A}^{n+1} \setminus \{0\} \to K$ by the same fla, regular on $\mathbb{A}^{n+1} \setminus \{0\}$

$\Rightarrow$ regular on $\mathbb{A}^n \Rightarrow \tilde{\varphi} \in K[x_0, \ldots, x_n]$ polynomial

But $\tilde{\varphi}(x_0 \ldots x_n) = \tilde{\varphi}(\lambda x_0, \ldots \lambda x_n) \Rightarrow \tilde{\varphi} = \text{const}$ (degree 0 homogeneous polynom)

Q: What about x_i? Are these functions on $\mathbb{P}^n$?

A: next time.