Lecture 15 | Line bundles

Def A <u>line bundle</u> on X is the following data:

① A map $E \xrightarrow{\pi} X$

② The structure of 1-dim vector space on each fiber $\pi^{-1}(x)$. That is, if $e_1, e_2 \in E$ and $\pi(e_1) = \pi(e_2) = x$ then we can define $\alpha e_1 + \beta e_2 \in \pi^{-1}(x)$ for $\alpha, \beta \in \mathbb{R}$

③ <u>Local triviality</u>: for each $x \in X$ there is an open subset $U \ni x$ such that
$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\sim"] \arrow[dr, "\pi"'] & & U \times \mathbb{A}^1 \arrow[dl] \\ & U & \end{tikzcd}$$
respecting vector space structure.

Ex $X = \mathbb{P}^n$, $E = \{(p, \ell) : p \in \ell\} \simeq Bl_0 \mathbb{A}^{n+1}$

$\pi : E \to \mathbb{P}^n \qquad \pi(p, \ell) = \ell$

This is called the <u>tautological line bundle</u> on $\mathbb{P}^n$ and denoted by $\mathcal{O}(-1)$.

Def More generally, we define:

<u>for $k \geq 0$</u>: $\mathcal{O}(k) = \{(f, \ell) : f$ is a degree k homogeneous polynomial on $\ell\}$

<u>for $k \leq 0$</u>: $\mathcal{O}(k) = \{(f, \ell) : f$ is a degree k homogeneous polynomial on $\ell^*\}$

<u>for $k = 0$</u>: $f = \text{const} \Rightarrow \mathcal{O}(0) \simeq \mathbb{A}^1 \times \mathbb{P}^n$ trivial bundle.

Lemma: $\mathcal{O}(k)$ is a line bundle on $\mathbb{P}^n$ for all $k \in \mathbb{Z}$.

Pf: We need to check local triviality

$U_i = \{x_i \neq 0\}$ $\quad v = \left(\frac{x_0}{x_i} : \dots : 1 : \dots \frac{x_n}{x_i}\right)$ = basis in ℓ

Any vector in $\ell = tv$, $t \in \mathbb{k}$

$(k \geq 0)$ f homog degree $k \Rightarrow f(tv) = \alpha t^k$

$$f(x_0, \dots, x_n) = f(x_i \cdot v) = \alpha \cdot x_i^k$$

Claim $\pi^{-1}(U) \simeq U \times \mathbb{A}^1$

$$(\alpha t^k, \ell) \longleftrightarrow (\ell, \alpha)$$

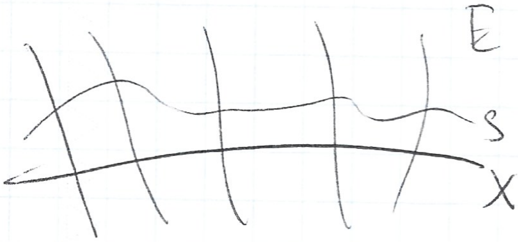
$$(f, \ell) \longrightarrow \left(\ell, \frac{f(x_0, \dots, x_n)}{x_i^k}\right)$$

For $k \leq 0$ we use the following fact w/o proof:

Fact Suppose V = 1-dim vector space over $\mathbb{k}$.
Then there is a canonical isomorphism

$$\{\text{degree } k \text{ polynomials on } V^*\} \longleftrightarrow \{\text{degree } (-k) \text{ Laurent polynomials on } V\}.$$

Def A section is a map $s: X \to E$ such that $\pi(s(x)) = x$



Ex: $E = X \times \mathbb{A}^1$ trivial bundle
section $\longleftrightarrow$ function $X \to \mathbb{A}^1$.

Lemma Assume E has a section $s: X \to E$ such that $s(x) \neq 0$ for all x. Then $E \simeq X \times \mathbb{A}^1$ is trivial.

Proof We define maps

$$\begin{tikzcd} E \arrow[rr, bend left, "\varphi_1"] \arrow[dr, "\pi"'] & & X \times \mathbb{A}^1 \arrow[ll, bend left, "\varphi_2"] \arrow[dl] \\ & X & \end{tikzcd}$$

$$\varphi_1(e) = \left(\pi(e), \frac{e}{s(\pi(e))}\right)$$

$$\varphi_2(x, \alpha) = \alpha \cdot s(x)$$

Since $s(x) \neq 0$, it is a basis vector in $\pi^{-1}(x) \simeq \mathbb{A}^1$.
Therefore any vector in $\pi^{-1}(x)$ can be written as $\alpha s(x)$.
Conversely, e and $s(\pi(e))$ live in the same 1-dim space, and $s(\pi(e)) \neq 0$, so $\frac{e}{s(\pi(e))} \in k$ is well defined.

$\square$

Lemma Suppose s_1, s_2 = two sections of E.
Then $\frac{s_1}{s_2}$ = rational function on X well defined on $\underbrace{\{s_2 \neq 0\}}_{\text{open}}$

Pf $s_1(x), s_2(x) \in \underset{\mathbb{A}^1}{\underset{\wr\wr}{\pi^{-1}(x)}} \Rightarrow \frac{s_1(x)}{s_2(x)}$ is well defined if $s_2(x) \neq 0$.

$\square$

Ex $E = \mathcal{O}(k)$ on $\mathbb{P}^n$, $k \geq 0$

F = degree k polynomial in $x_0, \ldots x_n$
= degree k polynomial on $\mathbb{A}^{n+1}$
$\rightsquigarrow$ section of $\mathcal{O}(k)$

$$s(\ell) = (F|_\ell, \ell)$$

degree k polynomial on ℓ.

Conclusion $x_0, \ldots x_n$ = NOT functions on $\mathbb{P}^n$

BUT sections of $\mathcal{O}(1)$; $x_0^5 x_1^7 x_2^3$ = section of $\mathcal{O}(5+7+3)$ $= \mathcal{O}(15)$.