Lecture 16 | $E \xrightarrow{\pi} X$ line bundle

<u>Def</u> $\Gamma(E) = H^0(X, E)$ = space of regular sections of E

<u>Note</u>: $s_1, s_2 : X \to E$ sections $\Rightarrow \alpha s_1 + \beta s_2$ is a section so $\Gamma(E)$ is a vector space.

<u>Recall</u> $X = \mathbb{P}^n$, $\mathcal{O}(k) = \{(f, \ell) : \ell \in \mathbb{P}^n,\ f = \text{degree } k \text{ homogeneous polynomial on } \ell\}$

<u>Thm</u> $\Gamma(\mathcal{O}(k)) = \begin{cases} \text{degree } k \text{ homogeneous poly in } x_0, \dots, x_n & k \geq 0 \\ 0, & k \leq 0 \end{cases}$

<u>Proof</u> $U_i = \{x_i \neq 0\}$ on U_i we have section x_i^k of $\mathcal{O}(k)$

Assume s = global ~~...~~ section of $\mathcal{O}(k)$

$$\frac{s}{x_i^k} = \text{function on } \mathbb{P}^n, \text{ regular on } U_i$$

$$\frac{s}{x_i^k} = f_i\left(\frac{x_0}{x_i}, \dots \frac{x_n}{x_i}\right)$$

$$s(x) = x_i^k f_i\left(\frac{x_0}{x_i}, \dots \frac{x_n}{x_i}\right) = x_i^k \frac{h_i(x_0, \dots x_n)}{x_i^d}$$

where $h_i(x_0, \dots x_d)$ = homogeneous degree d, coprime to x_i

On $U_i \cap U_j$ we have

$$s(x) = x_i^k \frac{h_i(x_0 \dots x_n)}{x_i^d} = x_j^k \frac{h_j(x_0 \dots x_n)}{x_j^d}$$

$$x_i^{k-d} h_i(x_0 \dots x_n) = x_j^{k-d}(x_0 \dots x_n). \quad (*)$$

<u>Case 1</u> $k < d$, then $x_i^{k-d} h_i$ has denominator x_i^{d-k}

which contradicts (*).

Case 2: $d \le k$, then $s(x) = x_i^{k-d} h_i(x_0 \dots x_n) =$ homogeneous polynomial of degree k.

Conversely, any homogeneous degree k poly in $x_0 \dots x_n$ defines a global section of $\mathcal{O}(k)$. $\boxed{\text{Cor } \Gamma(\mathcal{O}) = \text{functions on } \mathbb{P}^1 = \mathbb{k}}$ $\square$

Cor $\mathcal{O}(k) \not\cong \mathcal{O}(m)$ for $k \neq m$

Proof: Case 1 $k, m \ge 0$ $\dim \Gamma(\mathcal{O}(k)) = \binom{n+k}{k}$ = homog. degree k in $x_0 - x_n$

$$\neq$$

$\dim \Gamma(\mathcal{O}(m)) = \binom{n+m}{m}$

Case 2: $k \ge 0$, $m < 0$ $\dim \Gamma(\mathcal{O}(k)) > 0$, $\dim \Gamma(\mathcal{O}(m)) = 0$ $\Rightarrow \mathcal{O}(k) \not\cong \mathcal{O}(m)$

Case 3 $k, m < 0$ Assume $\mathcal{O}(k) \simeq \mathcal{O}(m)$

Then $\mathcal{O}(-k) = \mathcal{O}(k)^* = \mathcal{O}(m)^* = \mathcal{O}(-m)$

but $-k \neq -m \ge 0$ $\Rightarrow \mathcal{O}(-k) \not\cong \mathcal{O}(-m)$, contradiction

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Here we used the notion of a dual line bundle E^* with fibers $\pi^{-1}(x)^*$, and $\boxed{\mathcal{O}(k)^* = \mathcal{O}(-k)}$.

Lemma

$$\begin{array}{c} E \\ \downarrow \pi \end{array} \quad E = \text{line bundle on } Y.$$

$f: X \longrightarrow Y$ $\quad f^*E = \{(x, e) : x \in X, e \in E, f(x) = \pi(e)\}$

Then f^*E = line bundle on X

Proof
$$\begin{tikzcd} f^*E \arrow[r] \arrow[d, "\tilde{\pi}"'] & E \arrow[d, "\pi"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$
$(\tilde{\pi})^{-1}(x) = \{e : \pi(e) = f(x)\} = \pi^{-1}(f(x))$ // 1 d vector space.

Assume $\pi^{-1}(U) \simeq U \times \mathbb{A}^1$, $U \subset Y$

Then Define $\tilde{U} = f^{-1}(U)$ open in X $\quad \tilde{\pi}^{-1}(\tilde{U}) = \tilde{U} \times \mathbb{A}^1$ (check it!)

so f^*E is locally trivial $\square$.

Rmk If $s : Y \to E$ is a section, define

$f^*s : X \to f^*E$, $\quad f^*s(x) = (x, s(f(x))) \in f^*E$.

section of f^*E.

Thm There is a bijection

$$\left\{ \begin{array}{c} \text{regular maps} \\ X \xrightarrow{f} \mathbb{P}^n \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{line bundles } E \to X \\ + \text{ sections } s_0, \dots, s_n \\ \text{such that } s_i \neq 0 \text{ simultaneously} \end{array} \right\}$$

Pf ① Given a map $f : X \to \mathbb{P}^n$, define $E = f^*\mathcal{O}(1)$

$s_i = f^*x_i$. If $s_0(p) = \dots = s_n(p) = 0$ then

$x_0(f(p)) = \dots = x_n(f(p)) = 0$, contradiction.

② Given a line bundle E with sections $s_0, \dots, s_n$

define $f(x) = [s_0(x) : \dots : s_n(x)]$

What does it mean? Choose a basis e in $\pi^{-1}(x)$

$s_i(x) = \lambda_i e$ and $[s_0(x) : \quad : s_n(x)] = [\lambda_0 : \dots : \lambda_n]$

$e' = \mu e \Rightarrow [\frac{\lambda_0}{\mu} : \dots \frac{\lambda_n}{\mu}] \sim [\lambda_0 : \dots : \lambda_n]$. $\square$