Lecture 8 | More on projective space $\mathbb{P}^n$

$[x_0 : \dots : x_n]$ homogeneous coordinates

① Open charts $U_i = \{x_i \neq 0\}$ $[x_0 : \dots x_i : \dots x_n] \sim [\frac{x_0}{x_i} : \dots : 1 : \dots \frac{x_n}{x_i}]$

(HW1) [$\{x_i = 0\}$ closed since x_i homogeneous]

$U_i = \mathbb{A}^n$ with coordinates $\frac{x_0}{x_i}, \dots \widehat{\frac{x_i}{x_i}}, \dots \frac{x_n}{x_i}$

<u>Lemma</u> The induced Zariski topology on U_i agrees with Zariski topology on U_i, so U_i is <u>homeomorphic to $\mathbb{A}^n$</u>. (wrt Zariski top)

<u>Pf</u>: ① Assume $\{F_k = 0\} = Z$ closed in $\mathbb{P}^n$

F_k = homogeneous degree k polynomial

$Z \cap U_i = \{F_k(x_0, \dots x_n) = 0, x_i \neq 0\} =$

$= \{x_i^k F_k(\frac{x_0}{x_i}, \dots 1, \dots \frac{x_n}{x_i}) = 0, x_i \neq 0\} =$

$= \{\cancel{F} f(\frac{x_0}{x_i}, \dots 1, \dots \frac{x_n}{x_i}) = 0\} \subset U_i$ closed in $\mathbb{A}^n$.

$F(y_0, \dots 1, \dots y_n) = f(y_0, \dots \widehat{y_i}, \dots y_n)$.

Any closed is intersection of these.

② $Z' = \{f = 0\} \subset \mathbb{A}^n$

F = homogenization of f (HW1)

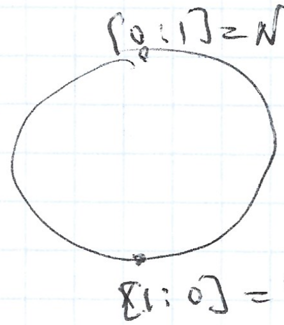
$\underset{Z''}{\{F = 0\}} \cap U_i = \{f = 0\}$

$\Rightarrow Z' = Z \cap U_i$ is closed in induced topology.

Note $\mathbb{P}^n = U_0 \cup U_1 \cup \dots \cup U_n$ open cover

Ex $\mathbb{CP}^1 \simeq S^2$

$[x_0 : x_1]$



$U_0 = \mathbb{CP}^1 \setminus [0:1] \simeq \mathbb{A}^1 = \mathbb{C}_z$

sphere minus north pole

$U_1 = \mathbb{CP}^1 \setminus [1:0] \simeq \mathbb{A}^1 = \mathbb{C}_w$

sphere minus south pole.

$z = \dfrac{x_1}{x_0}$ = coord on U_0

$w = \dfrac{x_0}{x_1}$ = coord on U_1 $\boxed{w = \frac{1}{z}}$ change of coords between the charts.

② $\mathbb{P}^n = \{x_0 \neq 0\} \cup \{x_0 = 0\} = U_0 \cup \mathbb{P}^{n-1} = \mathbb{A}^n \cup \mathbb{P}^{n-1}$

open

hyperplane at ∞

closed

By induction

$$\mathbb{P}^n = \mathbb{A}^n \sqcup \underbrace{\mathbb{A}^{n-1} \sqcup \mathbb{A}^{n-2} \sqcup \dots \sqcup pt}_{\mathbb{P}^{n-1}}$$

cell decomposition / affine paving

Concretely: $\mathbb{P}^n = \{x_0 \neq 0\} \sqcup \{x_0 = 0, x_1 \neq 0\} \sqcup \{x_0 = x_1 = 0, x_2 \neq 0\} \sqcup \dots$

look at first nonzero coord.

Cor $\# \mathbb{P}^n(\mathbb{F}_q) = q^n + q^{n-1} + \dots + 1$

$$= \frac{q^{n+1} - 1}{q - 1} = \frac{\#(\mathbb{A}^{n+1} - \{0\})}{\#(\mathbb{k}^\times)}$$

Aside: Def I = homogeneous ideal in $\mathbb{k}[x_0, \dots, x_n]$

if I is generated by homogeneous polynom.

Exercise. I homogeneous ideal $\Leftrightarrow$ $\forall$ $f = \sum f_i$ (homog. components)

~~Then~~ $f \in I$ iff $f_i \in I$.

Thm (projective Nullstellensatz) Let I be a homogeneous ideal in $\mathbb{K}[x_0, \dots x_n]$, $\mathbb{K}$ alg. closed. Then $Z(I) \subset \mathbb{P}^n$ is empty iff $(x_0, \dots x_n)^N \subset I$ for some N.

Proof • If $(x_0, \dots x_n)^N \subset I$ then any point in $Z(I)$ satisfies $x_0^N = \dots = x_n^N = 0 \Rightarrow x_0 = \dots = x_n = 0$ contradiction.

• Assume $Z(I) = \emptyset$, then $\tilde{Z} \subset \mathbb{A}^{n+1}$ is empty or $\{0\}$.

Case 1 $\tilde{Z} = \emptyset \Rightarrow$ by Nullstellensatz $I = \mathbb{K}[x_0, \dots x_n]$

Case 2 $\tilde{Z} = \{0\} \rightarrow$ by Nullstellensatz $\sqrt{I} = (x_0, \dots, x_n)$ so $x_i^{N_i} \in I$ and $(x_0, \dots x_n)^N \subset I$ for $N \geq \max(N_i)$. $\square$

Rings of functions: $X =$ alg. set in $\mathbb{A}^n$

Def A function $f : X \to \mathbb{K}$ is called ~~[illegible]~~ regular if there is $\tilde{f} \in \mathbb{K}[x_0, \dots x_n]$ such that $\tilde{f}|_X = f$.

$A(X) =$ ring of regular functions on X.

Lemma $A(X) = \dfrac{\mathbb{K}[x_1, \dots x_n]}{I(X)}$

Proof We have a homomorphism

$$\mathbb{K}[x_1, \dots x_n] \xrightarrow{\varphi} A(X)$$

$$\varphi(\tilde{f}) = \tilde{f}|_X$$

, surjective by definition.

$\ker \varphi = I(X) \Rightarrow$ by Isomorphism Theorem

$$A(X) \cong \frac{\mathbb{K}[x_1 \ldots x_n]}{I(X)}.$$

$\square$

Lemma (a) We have a bijection

$$\{\text{ideals in } A(X)\} \longleftrightarrow \{\text{ideals in } \mathbb{K}[x_1 \ldots x_n] \text{ containing } I(X)\}$$

(b) If $\mathbb{K}$ alg. closed:

$$\{\text{max. ideals in } A(X)\} \longleftrightarrow \{\text{max. ideals in } \mathbb{K}[x_1 \ldots x_n] \text{ containing } I(X)\} =$$

$$= \{(x_1 - a_1, \ldots x_n - a_n) \mid (a_1 \ldots a_n) \in X\} = \{\text{points in } X\}$$

(c) $A(X)$ is Noetherian.

Pf (a) is an exercise

(b) and (c) follow from (a).

Lemma $\{\text{closed subsets in } X\} \longleftrightarrow \{\text{ideals in } A(X)\}$

Pf (a) $Z \subset X$ closed, alg. set in $\mathbb{A}^n$

then $I(Z) \supset I(X)$, so $I(Z)/I(X) \subset$ ideal in $\frac{\mathbb{K}[x_1 \ldots x_n]}{I(X)}$

(b) $J' =$ ideal in $A(X) \longleftrightarrow J =$ ideal in $\mathbb{K}[x_1 \ldots x_n]$ containing $I(X)$

$J \supset I(X)$

$Z(J) \subset Z(I(X)) = X$