Lecture 9

X = alg. set in $\mathbb{A}^n$

$$A(X) = \frac{\mathbb{K}[x_1 \ldots x_n]}{I(X)} = \text{ring of regular functions on } X$$

$$\left\{ \begin{array}{c}\text{Ideals in} \\ A(X)\end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c}\text{Ideals in } \mathbb{K}[x_1 \ldots x_n] \\ \text{containing } I(X)\end{array} \right\}$$

Cor Closed subset in $X \Leftrightarrow$ ideals in $A(X)$

Pf $Z \subset X$ algebraic $\Leftrightarrow I(Z) \supset I(X)$.

Lemma (Very important) Assume $f \in A(X)$ and $f \neq 0$ in X. Then f is invertible in $A(X)$, that is, there is $g \in A(X)$ such that $fg = 1$.

$\mathbb{K}$ alg. closed

Pf Consider ideal $J = (f, I(X))$. Then

$Z(J) = \{f=0\} \cap X = \emptyset$, so by Nullstellensatz

$J = \mathbb{K}[x_1 \ldots x_n] \Rightarrow \alpha \cdot f + g = 1$ $\quad \alpha \in \mathbb{K}[x_1 \ldots x_n]$, $g \in I(X)$

$\Rightarrow \alpha f = 1$ in $A(X)$. $\square$

Morphisms $X \subset \mathbb{A}^n_{x_1 \ldots x_n}$, $Y \subset \mathbb{A}^m_{y_1 \ldots y_m}$ alg. sets

Def A *morphism* $\varphi : X \to Y$ is a function such that there exist polynomials $p_1(x_1 \ldots x_n), \ldots, p_m(x_1 \ldots x_n)$ such that

$$\varphi(x_1 \ldots x_n) = (p_1(x_1 \ldots x_n), \ldots, p_m(x_1 \ldots x_n))$$

$$y_i = p_i(x_1 \ldots x_n) \quad i = 1 \ldots m.$$

Ex A morphism $X \to \mathbb{A}^1$ is a regular function on X.

Ex A morphism $X \to \mathbb{A}^m$ is an m-tuple of regular functions on X.

$\boxed{\text{Note}}$ ~~Ex~~: A morphism $X \to Y \subset \mathbb{A}^m$ is an m-tuple of regular functions $p_1, \ldots, p_m : X \to \mathbb{A}^1$ such that

$$g(p_1(x), \ldots, p_m(x)) = 0 \text{ for all } x \in X$$

and $g \in I(Y)$.

Ex: $X = \mathbb{A}^1_t$, $Y = \{x^2 = y^3\} \subset \mathbb{A}^2$

$$\varphi(t) = (t^3, t^2)$$

Well defined since $(t^3)^2 = (t^2)^3 \Rightarrow \varphi(t) \in Y$ for all t.

Thm There is a bijection

$$\left\{\begin{matrix}\text{morphisms}\\ \varphi : X \to Y\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{algebra homomorphisms}\\ \varphi^* : A(Y) \to A(X)\end{matrix}\right\}.$$

Pf (1) Suppose $\varphi : X \to Y$ is a morphism, define

$$\varphi^* : A(Y) \to A(X), \quad \varphi^* g = g \circ \varphi \qquad \begin{tikzcd} X \arrow[r, "\varphi"] \arrow[rr, bend right, "g \circ \varphi"'] & Y \arrow[r, "g"] & \mathbb{A}^1 \end{tikzcd}.$$

If g is a polynomial in $y_1 \ldots y_m$ then $g(p_1(x_1 \ldots x_n), \ldots p_m(x_1 \ldots x_n))$ is a ~~[illegible]~~ polynomial in $x_1 \ldots x_n$, so $g \circ \varphi \in A(X)$. Also, is is easy to see that

$$\varphi^*(g_1 g_2) = \varphi^*(g_1) \cdot \varphi^*(g_2)$$

$$\varphi^*(g_1 + g_2) = \varphi^*(g_1) + \varphi^*(g_2)$$

So it is an algebra homomorphism.

(2) Assume $\alpha : A(Y) \to A(X)$ homomorphism.

$y_i \in A(Y) \Rightarrow \alpha(y_i) \in A(X)$ and we can find some polynomials $p_i(x_1 \ldots x_n)$ such that $p_i(x_1 \ldots x_n)\big|_X = \alpha(y_i)$. Now since α is a homomorphism we get $\alpha(g(y_1 \ldots y_m)) = g(\alpha(y_1), \ldots \alpha(y_m))$

$= g(p_1(x_1 \ldots x_n), \ldots p_m(x_1 \ldots x_n))$ on X.

Define $\varphi(x_1 \ldots x_n) = (p_1(x_1 \ldots x_n), \ldots p_m(x_1 \ldots x_n))$

then $\varphi^* = \alpha$ and we are done

Note: If $g \in I(Y)$ then $g = 0$ in $A(Y)$ and $\alpha(g) = 0$.

Therefore $g(p_1(x_1 \ldots x_n), \ldots p_m(x_1 \ldots x_n)) = 0$ on X

and the point $(p_1(x_1 \ldots x_n), \ldots p_m(x_1 \ldots x_n))$ is indeed in Y.

<u>Def</u> $\varphi : X \to Y$ isomorphism (X, Y = alg. sets)

if φ is a morphism, bijection and $\varphi^{-1} : Y \to X$ is a morphism.

<u>Lemma</u> φ is an isomorphism $\Leftrightarrow$ $\varphi^* : A(Y) \to A(X)$ is an algebra isomorphism.

<u>Ex</u> $\mathbb{A}^1_t \xrightarrow{\varphi} \{x^2 = y^3\} = Y \subset \mathbb{A}^2$

$\varphi(t) = (t^3, t^2)$

$\varphi^*(x) = t^3$, $\varphi^*(y) = t^2$ $\quad \varphi^*(g(x,y)) = g(t^3, t^2) \in \mathbb{K}[t]$

$\varphi^* A(Y) =$ subring of $\mathbb{K}[t]$ generated by t^2, t^3

$\Rightarrow \varphi^*$ is NOT an isomorphism.

Note that φ is a bijection: $\varphi^{-1}(x,y) = \begin{cases} 0, & \text{if } x=y=0 \\ x/y, & \text{if } y \neq 0 \end{cases}$

But φ^{-1} is not a morphism of algebraic sets.

__Lemma__ If $\varphi: X \to Y$ is a morphism then ~~[illegible]~~ φ is continuous in Zariski topology

__Pf__ Suppose $Z \subset Y$ closed, need to prove $\varphi^{-1}(Z)$ is closed. ~~[illegible]~~ We have:

$$x \in \varphi^{-1}(Z) \Leftrightarrow \varphi(x) \in Z \Leftrightarrow g(\varphi(x)) = 0$$

$$\text{for all } g \in I(Z) \Leftrightarrow \varphi^*(g)(x) = 0$$

So $\varphi^{-1}(Z) = Z(\underbrace{\varphi^* I(Z)}_{\text{not necessarily an ideal but it is OK.}})$

__Ex__ ① $X = \{y=0\} \subset \mathbb{A}^2$ ② $Y = \{y = x^2\} \subset \mathbb{A}^2$

$A(X) = \dfrac{\mathbb{K}[x,y]}{(y)} = \mathbb{K}[x]$ $\qquad A(Y) = \dfrac{\mathbb{K}[x,y]}{(y-x^2)}$

$A(X) \cong A(Y)$ $\qquad \dfrac{\mathbb{K}[x,y]}{(y-x^2)} \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \mathbb{K}[x]$

$\alpha(x) = x$
$\alpha(y) = x^2$ $\qquad \beta(x) = x$ $\qquad$ __Exercise__ $\alpha = \beta^{-1}$ isomorphism;

$\varphi_1 : X \longrightarrow Y \qquad \varphi_1(x,0) = (x, x^2) \qquad \varphi_1^* = \alpha$

$\varphi_2 : Y \longrightarrow X \qquad \varphi_2(x,y) = (x,0) \qquad \varphi_2^* = \beta$

$\varphi_1 \circ \varphi_2 = Id_Y \;,\quad \varphi_2 \circ \varphi_1 = Id_X \Rightarrow \boxed{X \simeq Y}$

③ $Z = \{y^3 = x^2\}$ $\quad A(Z) = \dfrac{\mathbb{K}[x,y]}{(y^3 - x^2)}$

$\varphi_3 : X \longrightarrow Z \qquad (x,0) \longrightarrow (x^3, x^2)$

$\varphi_3^* x = x^3 \qquad \varphi_3^* y = x^2.$