MAT 261A, Spring 2023
Homework 2

Due before 1:10 on Monday, April 17

Please write the homework solutions in connected sentences and explain your work. Mark the answers to each question. Scan or take pictures of your homework and upload it to Gradescope before due time.

1. Let V be the set of 2×2 complex matrices X such that $\mathrm{Tr}(X) = 0$ and $X^T = \overline{X}$.

a) Prove that V is a vector space over $\mathbb{R}$, find its basis and dimension. Is it a vector space over $\mathbb{C}$?

b) Consider the symmetric bilinear form $(X, Y) = \frac{1}{2}\mathrm{Tr}(XY)$ on V. Prove that it is positive definite.

2. Given a matrix $A \in SU(2)$ and X in space V from Problem 1, define

$$A(X) = AXA^*, \text{ where } A^* = \overline{A}^T.$$

a) Prove that $A(X) \in V$.

b) Prove that the formula above defines an action of $SU(2)$ on V preserving the bilinear form from (1b).

c) Use parts (a) and (b) to construct a homomorphism from $SU(2)$ to $SO(3)$.

3. Let G be a matrix Lie group and G_0 the connected component of identity in G.

a) Prove that G_0 is a normal subgroup of G.

b) Prove that x and y are in the same connected component of G if and only if $x^{-1}y \in G_0$.

c) Prove that all connected components of G are homeomorphic to G_0.

Note: you can use without proof that x and y are in the same connected component of G if and only if they are connected by a path.

4. Suppose that $GL(n, \mathbb{R})$ has a continuous representation V, that is, there is a continuous homomorphism $\rho : GL(n, \mathbb{R}) \to GL(V)$. Prove that the stabilizer of any vector $v \in V$ is a matrix Lie group.