

MAT 261A, Spring 2023  
Homework 3

**Due before 12:10 on Monday, April 24**

*Please write the homework solutions in connected sentences and explain your work. Mark the answers to each question. Scan or take pictures of your homework and upload it to Gradescope before due time.*

1. Find the exponents of the following matrices:

$$(a) \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \quad (b) \begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix}$$

2. Prove that

$$\exp \begin{pmatrix} 0 & -t \\ t & 0 \end{pmatrix} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

3. Suppose that  $\alpha$  is an **irrational** real number.

a) Prove that  $\Gamma_\alpha := \{e^{2\pi i \alpha k} : k \in \mathbb{Z}\}$  is a subgroup of  $U(1) = S^1$ .

b) Prove that  $\Gamma_\alpha$  contains elements of the form  $e^{2\pi i \phi}$  with  $0 < \phi < \frac{1}{N}$  for all  $N$ . *Hint: divide  $S^1$  into  $N$  equal arcs and prove that one of them contains at least two elements of  $\Gamma_\alpha$*

c) Use part (b) to prove that  $\Gamma_\alpha$  is dense in  $U(1)$  (and hence is NOT a matrix Lie group).

4. Suppose that  $\alpha$  is an **irrational** real number. Use problem 3 to prove that the one-parameter subgroup

$$\left\{ \exp \begin{pmatrix} 2\pi i t & 0 \\ 0 & 2\pi i \alpha t \end{pmatrix} : t \in \mathbb{R} \right\}$$

is dense in  $U(1) \times U(1)$  (and hence is NOT a matrix Lie group).