MAT 261A, Spring 2023
Homework 5
Due before 12:10 on Monday, May 8

Please write the homework solutions in connected sentences and explain your work. Mark the answers to each question. Scan or take pictures of your homework and upload it to Gradescope before due time.

Recall that the Lie algebra $\mathfrak{sl}_2$ has the basis E, F, H and relations
$$[H, E] = 2E, \ [H, F] = -2F, \ [E, F] = H.$$

1. Let $\lambda \in \mathbb{C}$. The *Verma module* $\Delta(\lambda)$ is an infinite-dimensional representation of $\mathfrak{sl}_2$ with basis $v_0 = v, v_1 = Fv, v_2 = F^2v, v_3 = F^3v, \ldots$ such that
$$Ev = 0 \text{ and } Hv = \lambda v$$

Clearly, $Fv_k = v_{k+1}$. Find Ev_k and Hv_k for all k.

2. Prove that $L(0)$ is trivial and $L(1)$ is isomorphic to the vector representation $\mathbb{C}^2$.

3. Prove that $L(2)$ is isomorphic to the adjoint representation of $\mathfrak{sl}_2$.

4. Suppose that $V = \oplus_m L(m)^{\oplus k_m}$, let a_i denote the dimension of the H-eigenspace in V with eigenvalue i. Prove that $a_i = a_{-i}$, $a_i \geq a_{i+2}$ for $i \geq 0$ and $a_i \geq a_{i-2}$ for $i \leq 0$.