MAT 261A, Spring 2023
Homework 7

Due before 1:10 on Monday, June 5

Please write the homework solutions in connected sentences and explain your work. Mark the answers to each question. Scan or take pictures of your homework and upload it to Gradescope before due time.

1. Every element in the Lie algebra $\mathfrak{so}(2n)$ can be presented as a $n \times n$ matrix of 2×2 blocks. Consider the following matrices:

$$H_a = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ at } (a,a) \text{ position on diagonal,}$$

$$C_1 = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}, C_2 = \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}, C_3 = \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}, C_4 = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix},$$

and for all $j < k$ define the matrices $E_{jk}^{(1)}, \ldots, E_{jk}^{(4)}$ by placing $C_1, \ldots C_4$ in block (j,k) and $-C_1^T, \ldots, -C_4^T$ in block (k,j).

a) Compute the commutators $[H_a, E_{jk}^{(1)}], \ldots, [H_a, E_{jk}^{(4)}]$ for all a, j, k.

b) Prove that $E_{jk}^{(1)}, \ldots, E_{jk}^{(4)}$ span the root subspaces (assuming that H_a span the Cartan subalgebra) and compute the corresponding roots.

c) Use (b) to conclude that, up to scaling, the roots of $\mathfrak{so}(2n)$ are the vectors $\pm(e_j + e_k), \pm(e_j - e_k)$ for all $j < k$. Here e_k is the standard basis in $\mathbb{R}^n$.

2. Let S_n be the space of degree n homogeneous polynomials in three variables x, y, z.

a) Prove that S_n is a representation of $\mathfrak{sl}(3)$.

b) Draw the weight diagram for S_n.

c) Prove that S_n is irreducible.

3. Use Weyl character formula to compute the character of the irreducible representation of $\mathfrak{sl}(3)$ with the highest weight $(3,2)$.

4. Recall that $\mathbb{C}^N \otimes \mathbb{C}^N = S^2\mathbb{C}^N \oplus \wedge^2\mathbb{C}^N$.

a) Compute the characters of $S^2\mathbb{C}^N$ and $\wedge^2\mathbb{C}^N$ as $\mathfrak{sl}(N)$ representations.

b) Prove that $S^2\mathbb{C}^N$ and $\wedge^2\mathbb{C}^N$ are irreducible.