Problem How many lines are there on a general cubic surface $X \subset \mathbb{P}^3$?

A: 27 How to compute this using Chern classes?

(1) $\{\text{lines } L \text{ in } \mathbb{P}^3\} \longleftrightarrow \{2\text{-planes } P_L \text{ in } \mathbb{A}^4\} = Gr(2,4)$

Define vector bundles on $Gr(2,4)$:

a) S = tautological vector bundle
fiber over $L = P_L$

b) V : fiber over L = degree 3 polynomials on P_L

Note: If s, t = coordinates on P_L, then

$$L = \{ s[x_0 : x_1 : x_2 : x_3] + t[y_0 : y_1 : y_2 : y_3] \}$$

for fixed 2 points $[x_0 : x_1 : x_2 : x_3]$ and $[y_0 : y_1 : y_2 : y_3]$

V_L = deg 3 polynomials on P_L =

$$= \text{Span}(s^3, s^2t, st^2, t^3) \qquad \dim V_L = 4$$

So V is a rank 4 bundle on $Gr(2,4)$

(2) We want to rephrase our problem in terms of V. Suppose F is the equation of the cubic surface X, define a section of

of the cubic surface X, define a section of V by $\sigma_F(L) = F|_L = F(s[x_0:x_1:x_2:x_3]+t[y_0:y_1:y_2:y_3])$

This is a cubic polynomial in s and t, so $\sigma_F(L) \in V_L$, $\boxed{\sigma_F \text{ is a section of } V}$

Also, $L \subset X \iff F|_L = 0 \iff \sigma_F(L) = 0$.

Conclusion

$\{\text{lines on } X\} \longleftrightarrow \left\{\begin{array}{c}\text{zero locus of a section} \\ \sigma_F \text{ of the vector bundle} \\ V \text{ on } Gr(2,4)\end{array}\right\}$

(3) Now we compute.

First of all, $\dim Gr(2,4) = 4$

$\dim\{\sigma_F = 0\} = \dim Gr(2,4) - \operatorname{rank}(V) = 4 - 4 = 0$

So for generic F $\{\sigma_F = 0\}$ is a finite number of points.

$\boxed{\#\text{points} = \text{Euler class of } V = c_4(V)}$

Second, $V_L = \mathrm{Sym}^3(P_L^*) = $ deg 3 polynomials on P_L

so $V = \mathrm{Sym}^3(S^*)$.

(4) We want to compute $\underset{c_4(\mathrm{Sym}^3(S^*))}{\underset{\shortparallel}{c_4(V)}}$ in terms of $c_i(S)$.

The formulas will be universal and natural in S, so by Splitting principle we can assume

$S = L_1 \oplus L_2 \qquad c_1(L_i) = t_i$

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$$S^* = L_1^* \oplus L_2^* \qquad c_1(L_i^*) = -t_i$$

$$\mathrm{Sym}^3(S^*) = \mathrm{Sym}^3(L_1^* \oplus L_2^*) =$$

$$= \underbrace{\left[(L_1^*)^3\right]}_{c_1 = -3t_1} \oplus \underbrace{\left[(L_1^*)^2 \otimes L_2^*\right]}_{c_1 = -2t_1 - t_2} \oplus \underbrace{\left[L_1^* \otimes (L_2^*)^2\right]}_{c_1 = -t_1 - 2t_2} \oplus \underbrace{\left[(L_2^*)^3\right]}_{c_1 = -3t_2}$$

By Whitney sum formula we get

$$c(\mathrm{Sym}^3(S^*)) = (1-3t_1)(1-2t_1-t_2)(1-t_1-2t_2)(1-3t_2)$$

$$c_4(\mathrm{Sym}^3(S^*)) = (-3t_1)(-2t_1-t_2)(-t_1-2t_2)(-3t_2)$$

$$= 9t_1t_2(2t_1+t_2)(t_1+t_2) =$$

$$= 9t_1t_2(2t_1^2 + 5t_1t_2 + 2t_2^2) =$$

$$= 9t_1t_2(2(t_1+t_2)^2 + t_1t_2)$$

$$= 9c_2(S)(2c_1^2(S) + c_2(S))$$

$$= \boxed{18\, c_2(S)c_1^2(S) + 9\, c_2^2(S).}$$

These formulas are universal and work for any S (rank 2)

(5) Next time, we will show that on $Gr(2,4)$

$$\langle c_1^2(S)c_2(S), [Gr(2,4)] \rangle = \langle c_2^2(S), [Gr(2,4)] \rangle = 1$$

and we get $\langle c_4(V), [Gr(2,4)] \rangle =$

$$= 18 \cdot 1 + 9 \cdot 1 = 27$$

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