Overview:

Big picture view: We've seen the axiomatic approach to Chern classes, and seen their recursive definition.

When the base is a smooth manifold, we can define Chern classes much more directly using connections & curvature.

Very informally:

- A <u>connection</u> is a way of identifying vectors in different fibers.
- To each connection is associated the <u>curvature</u>, choosing local coordinates, we can think of assigning a matrix of 2 forms to each point.
- Polynomials of a matrix which are invariant under choice of basis (eg trace, determinant) will let us extract invariants from the curvature - they won't depend on our choice of local coordinates.
- These invariants are differential forms, and don't depend on the choice of connection.
- We can construct the Chern classes, (or really their images in $H^*(X, \mathbb{C})$) in this way.

de Rham Cohomology

- Let X be a smooth, real manifold.
- A differential k-form is a section of $\Lambda^k T^* X$. (should remember from 239). Let $\Omega^k(X) = \{k\text{-forms}\}$.
- We can form a chain complex, the <u>de Rham complex</u>

$$\cdots \to \Omega^0(X) \xrightarrow{d} \Omega^1(X) \xrightarrow{d} \Omega^2(X) \to \cdots$$

where the differential map is the exterior derivative,

$$d(f\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}) = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} dx_j \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

The cohomology of this complex is the <u>de Rham cohomology</u>, $H^*_{dR}(X)$.

<u>Thm</u>: (de Rham's theorem)

$$H^*_{dR}(X) \cong H^*(X; \mathbb{R})$$

<u>Idea</u>: Show that we can replace $H_*(X; \mathbb{R})$ with $H_*^{smooth}(X; \mathbb{R})$, the smooth simplicial homology of X. Then H^*_{dR} pairs with H_*^{smooth} via integration of a k-form over a smooth k-cycle.

ex: $X = \mathbb{R}^2 - \{\vec{0}\}$, $H^*_{dR}(X)$

Pf: • A 0-form is a smooth function. A function with zero differential is locally constant, so

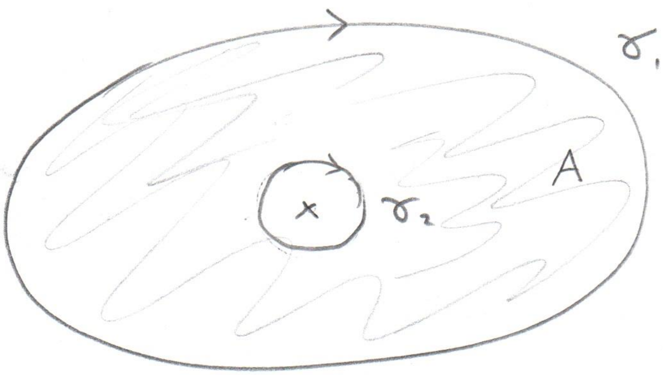
$$H^0_{dR}(X) = \mathbb{R}/\{0\} = \mathbb{R}.$$

• Any closed 1-form integrates to 0 around a loop away from the origin (Stokes). But we can have non-zero integral around 0, e.g. via

$$\alpha = \text{"}d\Theta\text{"} = \frac{y}{x^2+y^2}\,dx + \frac{-x}{x^2+y^2}\,dy,$$

which integrates to 2π along the unit circle.

However, the value when integrating β closed along any loop around the origin is the same:



$$\int_{\gamma_1}\alpha - \int_{\gamma_2}\alpha = \int_{\partial A}\alpha = \int_A d\alpha = \int_A 0 = 0.$$

• Exact 1-forms $\gamma = df$ integrate to 0 along any loop around the origin:

$$\int_{\text{loop}}\gamma = \int_{\text{loop}} df = \int_{\emptyset = \partial\,\text{loop}} df = 0.$$

• The converse is true: if γ integrates to 0 around the origin, γ is exact: fix a base point $p_0 \neq \vec{0}$. Then check:

$$f(p) = \int_{\substack{\text{any path from} \\ p_0 \text{ to } p}} \gamma$$ is well-defined, and a primitive for γ (i.e., $df = \gamma$).

So if β is closed, $\beta - \left(\frac{1}{2\pi}\int_{S^1}\beta\right)\alpha$ is exact, since it integrates to 0 around $\vec{0}$.

Thus $H^1_{dR}(X) \cong \mathbb{R}\cdot\alpha \cong \mathbb{R}$.

$H^2(X) = 0$:

Check: Given a 2-form $f(r,\theta)\, dr \wedge d\theta$ in polar coordinates,

$$\left(\int_1^r f(\rho,\theta)\, d\rho\right) d\theta$$ is a primitive.

So

$$H^k_{dR}(\mathbb{R}^2 - \{\vec{0}\}) = \begin{cases} \mathbb{R}, & k = 0,1 \\ 0, & \text{o/w} \end{cases}$$

We can also consider, e.g., k-forms with complex coefficients: $\Omega^k_{\mathbb{C}}(X) := \Omega^k(X) \otimes_{\mathbb{R}} \mathbb{C}$

Then $H^*_{dR}(X; \mathbb{C})$ is the cohomology of this complex, where $d(\alpha \otimes z) = d\alpha \otimes z$.

Connections:

How do we examine the infinitesimal change in a section of a vector bundle?

For a function $f: \mathbb{R}^m \to \mathbb{R}^n$, one can use the directional derivative:

$$D_x f(v) = \lim_{t \to 0} \frac{f(x+tv) - f(x)}{t}.$$

But for a section $\sigma: M \to E$ of a vector bundle $\begin{matrix} E \\ \pi \downarrow \\ M \end{matrix}$, $\sigma(x)$ and $\sigma(y)$ can live in different fibers, so it doesn't even make sense to subtract them.

On top of that, we only want to capture variation that comes from "actual" variation in the section, and not just from our choice of coordinates / basis.

(e.g. if you write a constant vector field $v = \begin{bmatrix} a \\ b \end{bmatrix}$ in $T\mathbb{R}^2$ with respect to the basis

$$e_1 = \begin{bmatrix} \cos x \\ \sin x \end{bmatrix}, \quad e_2 = \begin{bmatrix} -\sin x \\ \cos x \end{bmatrix},$$

you get $(a\cos x + b\sin x)e_1 + (-a\sin x + b\cos x)e_2$).

The right way to capture this information is a connection:

So, we can also regard a connection as a map

$$\nabla : \Gamma(E) \longrightarrow \Gamma(\underbrace{T^*_{\mathbb{C}}M}_{= T^*M \otimes_{\mathbb{R}} \mathbb{C}} \otimes_{\mathbb{C}} E)$$

We'll actually want to use the complexified version since we're looking at complex vector bundles.

You can think of RHS as "1-forms valued in E"

<u>Def:</u> A section σ is <u>parallel</u> if $\nabla_X \sigma = 0$ for any vector field X, ie, if $\nabla \sigma = 0$.

<u>Def:</u> On a trivial bundle $E \cong M \times \mathbb{R}^n$, we have global sections $e_1, \ldots, e_n$.

If we define these to be parallel, ie,

$$\nabla e_i = 0,$$

Then for any section $s = \sum_{i=1}^{n} h^i e_i$, we have

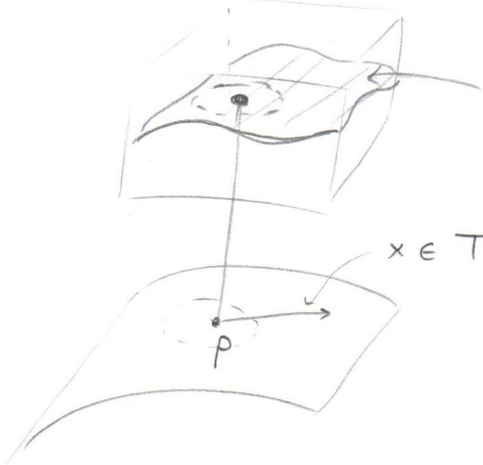
$$\begin{aligned} \nabla_X s = \nabla_X (\textstyle\sum h^i e_i) &= \textstyle\sum \nabla_X (h^i e_i) \\ &= \textstyle\sum X(h^i) e_i + h^i \nabla_X e_i \\ &= \textstyle\sum X(h^i) e_i, \end{aligned}$$

ie, all variation comes from the change in the h^i.

<u>Cor:</u> Every bundle has a connection

<u>Pf:</u> Check that any convex combination of connections is a connection.

Use a partition of unity subordinate to a trivializing cover, along with the trivial connection described above on each neighborhood.



I need the data of:

- A tangent vector ~~X at~~ each p
- The section $\Gamma(E)$, at least near each p.

And I output

- An element of the fiber $\pi^{-1}(p)$, at each p, ie., a new section.

So, a connection is a map

$$\nabla : \Gamma(TM) \times \Gamma(E) \longrightarrow \Gamma(E),$$

it should satisfy

- $\nabla_X \sigma$ is $C^\infty(M)$-linear in X
- (Leibniz) If $f \in C^\infty(M)$, then

$$\nabla_X f s = (Xf) s + f \nabla_X s$$
$$= df(X) s + f \nabla_X s.$$

ex: The usual directional derivative on $\mathbb{R}^n$ satisfies this. We can imagine suppressing the tangent vector part of the input - to each σ, we want to associate something that takes in tangent vector fields, and outputs sections.

$$\sigma \longmapsto \nabla_{\Box} \sigma$$

If we pick a local frame $e_1, \ldots, e_n$
(ie, a smooth choice of basis in each fiber in some neighborhood),
we can write (locally)

$$\nabla e_j = \underbrace{\sum_i \omega^i_j e_i}_{\text{really } \omega^i_j \otimes e_i} \quad \left(\text{or } \nabla_X e_j = \sum_i \omega^i_j(X) e_i\right) \quad (i)$$

Then we can piece the ω^i_j into a matrix,

$$\omega = [\omega^i_j].$$

We can rewrite (i) in matrix form as

$$[e_1, \ldots, e_n]\,\omega(X) = [\nabla_X e_1, \ldots, \nabla_X e_n]$$

or

$$[e_1, \ldots, e_n]\,\omega = [\nabla e_1, \ldots, \nabla e_n].$$

More generally, for $\sigma = \sum_j^n h^j e_j$, we have

$$\begin{aligned}
\nabla_X \sigma &= \sum_j \nabla_X h^j e_j \\
&= \sum_j \left(dh^j(X)\, e_j + \sum_i \omega^i_j(X)\, h^j e_i \right), \\
&= \sum_j dh^j(X) e_j + \sum_i e_i \sum_j \omega^i_j(X) h^j \\
&= \begin{bmatrix} dh^1(X) \\ \vdots \\ dh^n(X) \end{bmatrix} + \omega(X) \begin{bmatrix} h^1 \\ \vdots \\ h^n \end{bmatrix} \quad \text{(in matrix form)} \\
&= d\xi(X) + \omega(X)\,\xi
\end{aligned}$$

or $\nabla \xi = d\xi + \omega \xi$.

If we have another frame

$e_1', \dots, e_n'$

such that

$$e = e' \cdot a,$$

where $a : U \to GL(n; \mathbb{C})$ is a choice of matrix at each point of our neighborhood, then we have that the matrix ω' related to e' has:

$$\begin{aligned} e\omega &= \nabla e \\ &= \nabla(e'a) \\ &= (\nabla e')a + e'da \\ &= (e')\omega' a + ea^{-1}da \\ &= ea^{-1}\omega' a + ea^{-1}da. \end{aligned}$$

differential of a in each entry.

So $\omega = a^{-1}\omega' a + a^{-1}da$.

<u>Rmk:</u>

Curvature:

We can also define the curvature associated to a connection:

first, define a map $\overline{\nabla} : \Gamma(T^*_{\mathbb{C}}M \otimes E) \longrightarrow \underbrace{\Gamma(\Lambda^2 T^*_{\mathbb{C}} M \otimes E)}_{\text{2-forms valued in } E}$

satisfying

$$\overline{\nabla}(\Theta \otimes s) = d\Theta \otimes s - \Theta \wedge \nabla(s).$$

In a local frame $e_1, \dots, e_n$, can just use

$$\overline{\nabla}\left(\sum_i \Theta^i \otimes e_i\right) = \sum_i \left(d\Theta^i \otimes e_i - \Theta^i \wedge \nabla(s_i)\right).$$

We can compose this with ∇ to get a map

$$R : \Gamma(E) \xrightarrow{\nabla} \Gamma(T^*_{\mathbb{C}}M \otimes E) \xrightarrow{\overline{\nabla}} \Gamma(\Lambda^2 T^*_{\mathbb{C}} M \otimes E),$$

called the curvature tensor.

Fact: It is completely local: for each $x \in M$, $R(s)(x)$ depends only on $s(x)$, and not on nearby values of s.

So, R actually gives a smooth section of

$$\mathrm{Hom}(E, \Lambda^2 T^*_{\mathbb{C}} M \otimes E)$$

via $\quad s(x) \longmapsto K(s)(x).$

Pf: See Milnor

Fact: R is $C^\infty(M)$ linear in s.

In terms of a local frame, we can write

$$Re_j = \sum_i \Omega^i_j e_i,$$

again giving

$$[e_1, \ldots, e_n]\,\Omega = [Re_1, \ldots, Re_n], \text{ where } \Omega = [\Omega^i_j]$$

Our definition of $\overline{\nabla}$ yields

$$\begin{aligned}
Re_j &= \overline{\nabla}(\nabla e_j) \\
&= \overline{\nabla}\left(\sum_i \omega^i_j e_i\right) \\
&= \sum_i d\omega^i_j e_i - \omega^i_j \wedge \nabla e_i \\
&= \sum_i d\omega^i_j e_i - \omega^i_j \wedge \sum_k \omega^k_i e_k \\
&= \sum_i d\omega^i_j e_i - \sum_{i,k} \omega^i_j \wedge \omega^k_i e_k \\
&= \sum_i d\omega^i_j e_i - \sum_{k,i} \omega^k_j \wedge \omega^i_k e_i \\
&= \sum_i d\omega^i_j e_i + \sum_{i,k} \omega^i_k \wedge \omega^k_j e_i
\end{aligned}$$

In matrix form,

$$\Omega = d\omega + \omega \wedge \omega.$$

matrix wedge mult.

We also have that if $e = e' \cdot a$, then

$$\begin{aligned}
e\Omega &= R(e'a) \\
&= R(e')a \\
&= e'\Omega' a \\
&= ea^{-1}\Omega' a
\end{aligned}$$

So $\Omega = a^{-1}\Omega' a$.