Def A <u>vector bundle</u> is the following data:

- A topological space B (base)
- A top. space E (total space) + projection
  $\pi : E \longrightarrow B$
- The structure of a vector space on fibers $\pi^{-1}(b)$. That is, for $v_1, v_2 \in \pi^{-1}(b)$ one can define $\lambda v_1 + \mu v_2 \in \pi^{-1}(b)$

  We will always assume $\dim \pi^{-1}(b) = n$ is the same for all b.
- <u>Local triviality</u>: For $b \in B$ there is an open subset $U \ni b$ such that

$$\pi^{-1}(U) \xrightarrow[\sim]{h} U \times \mathbb{R}^n \quad \text{homeo}$$

and $h(\lambda v_1 + \mu v_2) = \lambda h(v_1) + \mu h(v_2)$

addition in $\mathbb{R}^n$

provided $\pi(v_1) = \pi(v_2)$.

<u>Examples</u> (1) <u>Trivial bundle</u> $E = B \times \mathbb{R}^n$

(2) M = smooth manifold

$E = TM$ = <u>tangent bundle</u> = $(p \in M, v \in T_p M)$

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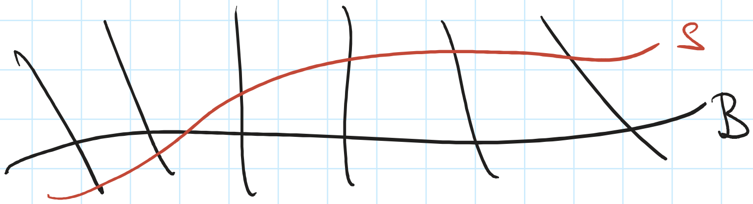
TM trivializes in coordinate charts. $\Big\downarrow \pi$

p

<u>Def</u> A <u>section</u> of E is a continuous map $s: B \to E$ such that $\pi(s(b)) = b$ for all b.

In other words, we choose a vector $s(b)$ in $\pi^{-1}(b)$



<u>Ex</u> ① E arbitrary, $s = 0$ zero section

(makes sense since we're in a vector space)

② $E = B \times \mathbb{R}^n$

$s(b) = (s_1(b), \ldots s_n(b))$ = n functions $B \to \mathbb{R}$.

③ $E = TM$ <u>tangent bundle</u>

section = <u>vector field</u> = tangent vector at each point of M

<u>Thm</u> $E \xrightarrow{\pi} B$ rank n is trivial if and only if one can find n sections $s_1, \ldots, s_n$ such that $s_1(b), \ldots s_n(b)$ are linearly independent.

such that $s_1(b), \dots s_n(b)$ are linearly independent at all points b. ($\Leftrightarrow$ basis in $\pi^{-1}(b)$)

Pf • Assume $E = B \times \mathbb{R}^n$, pick $s_i = (0 \dots \underset{i}{1} \dots 0)$

• Assume $s_1(b), \dots s_n(b)$ independent, get a map $B \times \mathbb{R}^n \longrightarrow E$

$$(b, \lambda_1 \dots \lambda_n) \longrightarrow \lambda_1 s_1(b) + \dots + \lambda_n s_n(b)$$

Exercise This is an isomorphism of vector bundles. ∎

Ex • TS^1 is trivial

$$b = (x, y) \quad v = (y, -x) \qquad v \in T_b S^1$$

continuous, $v \neq 0 \Rightarrow$ basis in TS^1

• TS^2 is not trivial

Fact (proof later) Any vector field on S^2 has at least one zero.

Assume s_1, s_2 = basis of sections of TS^2

Then s_1, s_2 indep. at all points of TS^2

$\Rightarrow s_1 \neq 0, s_2 \neq 0$. Contradiction. □

• Fact TS^3 is trivial (hint: $S^3 \simeq SU(2)$)

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Hard problem: For which n, TS^n is trivial?

$n = 1, 3, 7$ and that's all!

Def $M \subset \mathbb{R}^n$, **normal bundle** $= TM^{\perp}$ wrt usual dot product

Ex Normal bundle to $S^n_{\subset \mathbb{R}^{n+1}}$ is trivial for all n.

Pf $b = (x_0 \ldots x_n)$ $\quad v = (x_0 \ldots x_n) \perp T_b S^n$

nowhere vanishing section.

Def Tautological bundle on $\mathbb{R}P^n / \mathbb{C}P^n$

$\mathbb{R}P^n = \{\text{lines in } \mathbb{R}^{n+1} \text{ through } 0\}$ $\quad p \in \mathbb{R}P^n$, $p \leftrightarrow l_p = \text{line}$

$$E = \{(\underset{\mathbb{R}P^n}{\underset{\cap}{p}}, \underset{\mathbb{R}^n}{\underset{\cap}{v}}) : v \in l_p\}$$

$$E \xrightarrow{\pi} \mathbb{R}P^n \qquad \pi(p, v) = p$$

Clearly, $\pi^{-1}(p) = l_p \Rightarrow$ <u>line bundle</u>.

Thm Tautological bundle on $\mathbb{R}P^n$ is not trivial

Pf $\mathbb{R}P^n = S^n / \{\pm 1\}$ $\qquad E = \{p \in S^n/_{\pm 1} ; v \in l_p\}$

Suppose that there is a section $s(p) \in l_p$

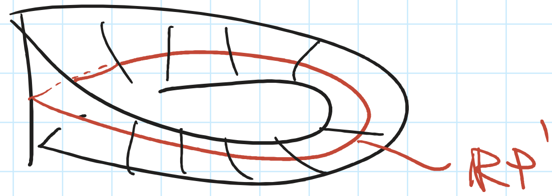
Suppose that there is a section $s(p) \in \ell_p$

Then $s(p) = t(p) \cdot p$ for some $t(p) \in \mathbb{R}$

$s(-p) = s(p)$ since $p \sim -p$

$t(-p) = -t(p) \Rightarrow$ by intermediate value theorem $t(p) = 0$ somewhere. $\square$

<u>Ex</u> Tautological bundle on $\mathbb{RP}^1 = S^1$ is the Möbius band



<u>More examples</u>

- E_1, E_2 vector bundles $\Rightarrow E_1 \oplus E_2,\ E_1 \otimes E_2$
- E^*, $\mathrm{Sym}^k E$, $\Lambda^k E$ etc.

<u>Ex</u> $(TM)^* = \Omega^1 M$ 1-forms on M

$$\Lambda^k(\Omega^1 M) = \Omega^k M \quad k\text{-forms on } M$$

- $E \subset F$ subbundle $\Rightarrow F/E$

- Warning

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, "\pi_E"'] & & F \arrow[dl, "\pi_F"] \\ & B & \end{tikzcd}$$

$\ker f$, $\mathrm{Im}\, f$ are usually <u>not</u> vector bundles

$$\begin{array}{c} E \quad\quad F \\ {\scriptstyle \pi_E}\searrow \;\; \swarrow{\scriptstyle \pi_F} \\ B \end{array}$$ usually **not** vector bundles

$$f_b : \pi_E^{-1}(b) \longrightarrow \pi_F^{-1}(b)$$

Rank of f_b might depend on b!

Def $M \subset K$ smooth manifolds.

The normal bundle to M in K is T_bK / T_bM, $b \in M$

Rank $= \dim K - \dim M$.

- $\begin{array}{ccc} & & E \\ & & \downarrow \\ B_1 & \xrightarrow{f} & B_2 \end{array}$ $\rightsquigarrow f^* E$

$f^*E = (b \in B_1, v \in E)$ such that $f(b) = \pi(v)$

Exercise This is a vector bundle on B_1.

Def A metric/Euclidean structure on E is a positive definite $\underbrace{\ }_{\text{symmetric}}$ bilinear form $(v_1, v_2) : \pi^{-1}(b) \to \mathbb{R}$

$(v, v) > 0$

HW Any vector bundle admits a continuous metric.

Cor E = Euclidean vector bundle, $F \subset E$

Usr E = Euclidean vector bundle, $F \subset E$

$\Rightarrow$ can define $F^\perp \subset E$

$$F \oplus F^\perp \cong E. \qquad F^\perp \simeq E/F$$

Warning Over $\mathbb{C}$, need Hermitian metrics, more complicated. In AG, E/F is well defined but $F^\perp$ is not.