Lecture 1/9

Tuesday, January 9, 2024 7:55 AM

# Obstruction theory

**Problem** Classify real/complex line bundles on a CW complex $X$.

**Recall**: Real line bdles $\longleftrightarrow$ maps $X \xrightarrow{f} \mathbb{R}P^\infty = Gr(1,\infty)$

$f^*\gamma^1$ = line bdle on $X$

Complex line bdles $\longleftrightarrow$ maps $X \to \mathbb{C}P^\infty$

How to classify such maps?

**Ex** $X = S^i$, need to compute $\pi_i(\mathbb{R}P^\infty)$, $\pi_i(\mathbb{C}P^\infty)$.

① $S^n \longrightarrow \mathbb{R}P^n$ 2:1 cover

$\pi_i(S^n) = \pi_i(\mathbb{R}P^n)$ $i \geq 2$ $\qquad \pi_i(\mathbb{R}P^n) = 0$

$\pi_1(\mathbb{R}P^n) = \mathbb{Z}_2$ $\qquad 2 \leq i \leq n-1$.

$$\pi_i(\mathbb{R}P^\infty) = \begin{cases} \mathbb{Z}_2, & i=1 \\ 0, & i>1 \end{cases}$$

② $S^{2n+1} \xrightarrow{S^1} \mathbb{C}P^n$

LES of fibration $\Rightarrow$ $\pi_1(\mathbb{C}P^n) = 0$

$\pi_2(\mathbb{C}P^n) = \pi_1(S^1) = \mathbb{Z}$

$\pi_i(\mathbb{C}P^n) = \pi_i(S^{2n+1})$, $i > 2$.

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In particular, $\pi_i(\mathbb{C}P^n) = 0$ for $3 \le i \le 2n$.

$$\pi_i(\mathbb{C}P^\infty) = \begin{cases} \mathbb{Z}, & i = 2 \\ 0, & \text{otherwise.} \end{cases}$$

Aside The Eilenberg-Maclane space $K(G, n)$ is characterized by $\pi_i(K(G,n)) = \begin{cases} G, & i = n \\ 0, & \text{otherwise} \end{cases}$

$\mathbb{R}P^\infty = K(\mathbb{Z}_2, 1)$ $\quad \mathbb{C}P^\infty = K(\mathbb{Z}, 2)$.

Thm $\text{Maps}(X, \mathbb{R}P^\infty)/_\sim = [X, \mathbb{R}P^\infty] = H^1(X, \mathbb{Z}_2)$

$[X, \mathbb{C}P^\infty] = H^2(X, \mathbb{Z})$

More generally, $[X, K(G,n)] = H^n(X; G)$.

Sketch of proof Say, for $\mathbb{C}P^\infty$, given $\alpha \in H^2(X, \mathbb{Z})$ we want to define a map $X \to \mathbb{C}P^\infty$.

0, 1 - cells $\longrightarrow$ pt.

2 - cell $\sigma \rightsquigarrow \alpha(\sigma) \in \mathbb{Z} = \pi_2(\mathbb{C}P^\infty) \Rightarrow$ get a map $\sigma \xrightarrow{f} \mathbb{C}P^\infty$.

3 - cell $\sigma^{(3)}$: $\partial \sigma^{(3)} = \sum \pm \sigma_i$

$$\alpha(\partial \sigma^{(3)}) = d\alpha(\sigma^{(3)}) = 0 = \sum \pm \alpha(\sigma_i) \in \pi_2(\mathbb{C}P^\infty)$$

$$\sum \pm \alpha(\sigma_i) = f(\partial \sigma^{(3)})$$

So $f|_{\partial \sigma^{(3)}}$ is null-homotopic

$\Rightarrow$ extends to $\sigma^{(3)}$.

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$k$-cell $\sigma^{(k)}$ for $k > 3$: $\partial\sigma^{(k)} = S^{k-1} \xrightarrow{f} \mathbb{C}P^\infty$
already constructed

Since $\pi_{k-1}(\mathbb{C}P^\infty) = 0$, $f|_{\partial\sigma^{(k)}}$ is null-homotopic

$\Rightarrow$ extends to $\sigma^{(k)}$. ∎

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<u>Conclusion</u>

$$\left\{\begin{array}{c}\text{real line bdles} \\ \text{on } X\end{array}\right\}_{\xi} \longleftrightarrow [X; \mathbb{R}P^\infty]_f$$

$$H^1(X, \mathbb{Z}_2)$$

$w_1(\xi) = f^* w_1(\gamma^1) = f^*(a)$

$a$ = generator of $H^1(\mathbb{R}P^\infty) \simeq \pi_1(\mathbb{R}P^\infty)$.

$$\left\{\begin{array}{c}\text{cpx line bdles} \\ \text{on } X\end{array}\right\} \longleftrightarrow [X, \mathbb{C}P^\infty]$$

$$H^2(X; \mathbb{Z})$$

$c_1(\xi) = f^* c_1(\gamma^1) = f^*(b)$

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What about higher rank bundles?

$V_k(\mathbb{R}^n) = \left\{\begin{array}{c}k\text{-tuples of} \\ \text{orthonormal vectors} \\ \text{in } \mathbb{R}^n\end{array}\right\}$ Stiefel manifold.

<u>Thm</u> $\pi_i(V_k(\mathbb{R}^n)) = 0$, $i < n-k$

$$\pi_{n-k}(V_k(\mathbb{R}^n)) = \begin{cases} \mathbb{Z}, & n-k \text{ even or } k=1 \\ \mathbb{Z}_2, & n-k \text{ odd and } k>1. \end{cases}$$

**Pf** For $k=1$, $V_k(\mathbb{R}^n) = S^{n-1}$ and the result is clear.

For $k>1$, we have a map $V_k(\mathbb{R}^n) \longrightarrow S^{n-1}$

$$\{v_1 \ldots v_k\} \longrightarrow v_1$$

$\{v_2, \ldots v_k\} = (k-1)$ orthonormal vectors in $v_1^\perp = \mathbb{R}^{n-1}$

So we get a locally trivial fibration w. fiber $V_{k-1}(\mathbb{R}^{n-1})$.

$$\to \pi_{i+1}(S^{n-1}) \to \pi_i(V_{k-1}(\mathbb{R}^{n-1})) \to \pi_i(V_k(\mathbb{R}^n)) \to \pi_i(S^{n-1})$$

$\pi_{i+1}(S^{n-1}) = 0 \leftarrow$ if $k>2$, $\pi_i(S^{n-1}) = 0$

for $i \leq n-k < n-1$

So $\pi_i(V_{k-1}(\mathbb{R}^{n-1})) \simeq \pi_i(V_k(\mathbb{R}^n))$

Base case: $\pi_i(V_2(\mathbb{R}^n)) = ?$

$$V_2(\mathbb{R}^3) = SO(3) \simeq \mathbb{R}P^3$$

$$\pi_1(V_2(\mathbb{R}^3)) = \mathbb{Z}_2.$$

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**Thm** $V_k(\mathbb{C}^n) = \{$hermitian orthonormal $k$-tuples$\}$.

$$\pi_i(V_k(\mathbb{C}^n)) = \begin{cases} 0, & i \leq 2(n-k) \\ \mathbb{Z}, & i = 2(n-k)+1. \end{cases}$$

**Ex** $V_1(\mathbb{C}^n) = S^{2n-1}$

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Now we want to define SW/Chern classes as obstructions.

$\xi \longrightarrow X$ rank $n$ vector bundle

Want to find $k$ linearly independent sections.

Use metric $\Rightarrow$ $k$ orthonormal sections.

- Can assume only one 0-cell of $X$
  define sections there arbitrarily.
- Given an $j$-cell $\sigma^{(j)}$, assume we defined $k$ section on $\partial\sigma^{(j)} \simeq S^{j-1}$

Then we get a map $\partial\sigma^{(j)} \xrightarrow{s} V_k(\mathbb{R}^n)$

If $j-1 < n-k$, $\pi_{j-1}(V_k(\mathbb{R}^n)) = 0 \Rightarrow$ . .

$s|_{\partial\sigma^{(j)}}$ is null-homotopic $\Rightarrow$ can extend to $\sigma^{(j)}$

- If $j-1 = n-k$, get <u>obstruction class</u>

$$\alpha(\sigma^{(j)}) = [s|_{\partial\sigma^{(j)}}] \in \pi_{j-1}(V_k(\mathbb{R}^n)) = \pi_{n-k}(V_k(\mathbb{R}^n))$$

$$\alpha \in H^{n-k+1}(X; \pi_{n-k}(V_k(\mathbb{R}^n)))$$

$$\downarrow$$

$H^{n-k+1}(X; \ldots)$

$$H^{n-k+1}(X; \mathbb{Z}_2)$$

<u>Thus</u> the image of $\alpha$ agrees with $w_{n-k+1}(\xi)$!