Review of cohomology

More info: Hatcher, Chapter 3.

① **Definition** $\to C_i(X,G) \xrightarrow{\partial} C_{i-1}(X,G) \to \dots$

Your favourite (cellular, singular, simplicial...) chain complex computing $H_*(X,G)$ with coefficients in G. Define

$\partial^2=0$

$$C^i(X,G) = C_i(X,G)^\vee = \mathrm{Hom}(C_i(X,G),\ G)$$

$$\to C^{i-1}(X,G) \xrightarrow{\partial^*} C^i(X,G) \to \dots$$

$(\partial^*)^2 = (\partial^2)^* = 0 \Rightarrow$ complex

$H^i(X,G) =$ homology of $(C^i(X,G), \partial^*) =$ cohomology of (X,G).

Ex $X = \mathbb{R}P^2$, $G = \mathbb{Z}$ coef

$$\begin{array}{ccccc} C_2 & \to & C_1 & \to & C_0 \\ \| & & \| & & \| \\ \mathbb{Z} & \xrightarrow{2} & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} \end{array}$$

$H_2 = 0 \quad H_1 = \mathbb{Z}_2 \quad H_0 = \mathbb{Z}$

$$\begin{array}{ccccc} C^2 & \leftarrow & C^1 & \leftarrow & C^0 \\ \| & & \| & & \| \\ \mathbb{Z} & \xleftarrow{2} & \mathbb{Z} & \xleftarrow{0} & \mathbb{Z} \end{array}$$

$H^2 = \mathbb{Z}_2 \quad H^1 = 0 \quad H^0 = \mathbb{Z}$

Ex $X = \mathbb{R}P^4$, $G = \mathbb{Z}_2$ coef

$C_*: \mathbb{Z}_2 \xrightarrow{0} \mathbb{Z}_2 \to \dots \xrightarrow{2} \mathbb{Z}_2 \xrightarrow{0} \mathbb{Z}_2$

Dual:

$C^*: \mathbb{Z}_2 \xleftarrow{0} \mathbb{Z}_2 \xleftarrow{0} \dots \leftarrow \mathbb{Z}_2$

$C_*: \mathbb{Z}_2 \xrightarrow{0} \mathbb{Z}_2 \to \dots \xrightarrow{\mathbb{Z}_2} \mathbb{Z}_2 \xrightarrow{0} \mathbb{Z}_2$ Dual: $C^*: \mathbb{Z}_2 \xleftarrow{0} \mathbb{Z}_2 \xleftarrow{0} \dots \leftarrow \mathbb{Z}_2$

$2=0$ $H_i(X, \mathbb{Z}_2) \simeq \mathbb{Z}_2$ $\boxed{H^i(X, \mathbb{Z}_2) = \mathbb{Z}_2}$

Ex $X = \mathbb{CP}^n$, $G = \mathbb{Z}$ even dimensional cells $i = 0, 1, \dots n$

$C_*: \mathbb{Z} \to 0 \to \mathbb{Z} \to 0 \to \dots \to \mathbb{Z} \to 0 \to \mathbb{Z}$

$C^*: \mathbb{Z} \leftarrow 0 \leftarrow \mathbb{Z} \leftarrow 0 \leftarrow \dots \mathbb{Z} \leftarrow 0 \leftarrow \mathbb{Z}$

$$H^i(\mathbb{CP}^n, \mathbb{Z}) = \begin{cases} \mathbb{Z}, & i = 0, 2, 4, \dots 2n \\ 0, & \text{otherwise}. \end{cases}$$

② Properties:

- Functoriality $X \xrightarrow{f} Y$ $f^*: H^i(Y) \to H^i(X)$
- Cup product: $\alpha \in H^i(X)$, $\beta \in H^j(X)$

$$\alpha \cup \beta \in H^{i+j}(X)$$

Associative, supercommutative

$$\beta \cup \alpha = (-1)^{ij} \alpha \cup \beta.$$

Functorial: $f^*(\alpha \cup \beta) = f^*\alpha \cup f^*\beta$.

- Poincaré duality

v1: X = smooth n-dimensional compact manifold

$$PD: H^i(X, \mathbb{Z}_2) \xrightarrow{\sim} H_{n-i}(X, \mathbb{Z}_2)$$

$PD: H^i(X, \mathbb{Z}_2) \longrightarrow H_{n-i}(X, \mathbb{Z}_2)$

v2: all of the above + oriented

$PD: H^i(X, \mathbb{Z}) \xrightarrow{\sim} H_{n-i}(X, \mathbb{Z})$.

If X is connected then $H^n(X, \mathbb{Z}_2) \simeq H_0(X, \mathbb{Z}_2) \simeq \mathbb{Z}_2$

If X is connected and oriented then $H^n(X, \mathbb{Z}) \simeq H_0(X, \mathbb{Z}) \simeq \mathbb{Z}$.

Fundamental class $[X] \in H_n(X, \mathbb{Z})$ if oriented

$H_n(X, \mathbb{Z}_2)$ any.

$1 \in H^0(X) \xleftrightarrow{PD} [X] \in H_n(X)$

(3) Computations X smooth, $\dim X = n$

$Y \subset X$ smooth subvariety $\dim Y = d_1$

$Z \subset$ $\dim Z = d_2$

$i_Y : Y \longrightarrow X$

$[Y] = i_{Y*}[Y] \in H_{d_1}(X)$ fund. class of Y

$[Z] \in H_{d_2}(X)$ $\alpha = PD[Y] \in H^{n-d_1}(X)$

($\mathbb{Z}_2$ coef / $\mathbb{Z}$ if orientable) $\beta = PD[Z] \in H^{n-d_2}(X)$

Fact Assume Y and Z are transversal. Then

$\alpha \cup \beta = PD[Y \cap Z]$

$\alpha \cup \beta = PD[Y \cap Z]$

$d_1 + d_2 - n$

Thm (a) $H^*(\mathbb{R}P^n, \mathbb{Z}_2) \simeq \mathbb{Z}_2[a] / (a^{n+1} = 0)$ $\quad a \in H^1$

as a ring

(b) $H^*(\mathbb{C}P^n, \mathbb{Z}) \simeq \mathbb{Z}[b] / (b^{n+1} = 0)$ $\quad b \in H^2$.

Pf: We do (b), (a) is similar.

$$H^{2i}(\mathbb{C}P^n, \mathbb{Z}) \overset{PD}{\simeq} H_{2n-2i}(\mathbb{C}P^n, \mathbb{Z}) \simeq \mathbb{Z}$$

$PD[Y_{n-i}] \longleftrightarrow$ spanned by $(n-i)$ plane Y_{n-i}

$$\underset{\langle b \rangle}{\overset{\shortparallel}{H^2}} \longleftrightarrow H_{2n-2}(\mathbb{C}P^n, \mathbb{Z}) \text{ ~~spanned~~ spanned by a hyperplane } H$$

$$b = PD[H]$$

$$b^i = b \cup \ldots \cup b = PD(H \cap H \cap \ldots \cap H) =$$

not transverse, perturb

$$= PD(H_1 \cap \ldots \cap H_i) =$$

i transverse hyperplanes

$$= PD([Y_{n-i}]) = \text{generator of } H^{2i}$$

So $b^i \neq 0$ for $i = 0, \ldots n$ and generates $H^{2i}(\mathbb{C}P^n)$. $\blacksquare$
