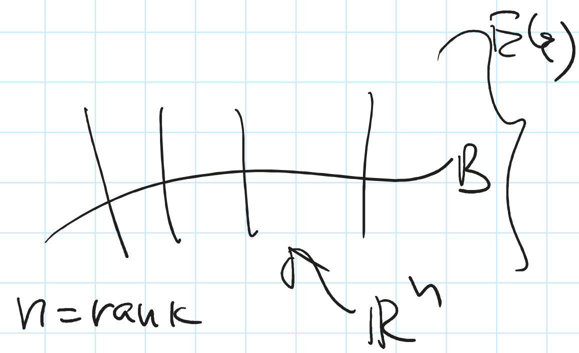
Lecture 3 (10/17)

Tuesday, October 17, 2023 11:01 AM

Chapter 4 (Melissa Zhang)

Stiefel-Whitney classes

$\xi : E \to B$



$\xi : E(\xi) \to B(\xi)$

Axiom 1

$w_i(\xi) \in H^i(B(\xi), \mathbb{F}_2)$

$w_i = 0$ if $i > \operatorname{rank} \xi$

$w_0(\xi) = 1$.

Axiom 2 If $f : B(\xi) \to B(\eta)$ is covered by some $\tilde{f} : \xi \to \eta$

$$\begin{tikzcd} E(\xi) \arrow[r, "\tilde{f}"] \arrow[d, "\xi"] & E(\eta) \arrow[d, "\eta"] \\ B(\xi) \arrow[r, "f"] & B(\eta) \end{tikzcd}$$

$w_i(\xi) = f^*(w_i(\eta))$

$\boxed{\xi = f^*\eta}$

Axiom 3 (Whitney product formula)

$w(\xi \oplus \eta) = w(\xi)\, w(\eta)$

$\{w_i(\xi) \in H^i(B, \mathbb{Z}_2)\}$

$\updownarrow$

total SW class $\to$ $w(\xi) = w_0 + w_1 + \ldots \in H^*(B)$

Unpack: $(w_0(\xi) + w_1(\xi) + \ldots)(w_0(\eta) + w_1(\eta) + \ldots)$

$w_k(\xi \oplus \eta) = \sum_{i+j=k} w_i(\xi)\, w_j(\eta)$

Axiom 4 (normalization)

Axiom 4 (normalization)

$w_1(\gamma_1^1) \neq 0$

γ_1^1 — tautological line bundle over $\mathbb{RP}^1 = P^1$

= Möbius band.

Immediate consequences ① $\xi \simeq \eta$ then $w_i(\xi) = w_i(\eta)$

(use Axiom 2)

② $\varepsilon : E \to B$ trivial vector bundle

$E = \mathbb{R}^r \times B$

$w_i(\varepsilon) = 0 \quad \forall i > 0$

$$\begin{array}{ccc} B\times\mathbb{R}^r & \longrightarrow & \mathbb{R}^r \\ \downarrow & & \downarrow \\ B & \longrightarrow & \{pt\} \end{array}$$

$H^i(pt) = 0$ for $i > 0$.

$\Rightarrow w_i(\mathbb{R}^r, *) = 0$

$\Rightarrow$ by Axiom 2 $w_i(B \times \mathbb{R}^r) = 0$.

③ ε trivial $\quad w_i(\varepsilon \oplus \eta) = w_i(\eta)$

$$w(\varepsilon \oplus \eta) \underset{\text{Axiom 3}}{=} w(\varepsilon)\cdot w(\eta) = 1 \cdot w(\eta) = w(\eta)$$

Algebraic aside

Fact In power series $\{1 + c_1 x + c_2 x^2 + \dots\}$ are the invertible elements.

Cor $w_0(\xi)=1 \Rightarrow$ we can invert total w's.

$$1-a^n=(1-a)(1+a+\dots+a^{n-1})$$

$$\frac{1-a^n}{1-a}=1+a+\dots+a^{n-1}$$

Observation $\xi\oplus\eta=\varepsilon$

ex. $M\subset\mathbb{R}^n$ $\quad \underbrace{TM}_{\text{tangent}}\oplus\underbrace{NM}_{\text{normal}}=\varepsilon$

$$w(\xi)w(\eta)=w(\varepsilon)=1$$

$$\Rightarrow w(\eta)=\frac{1}{w(\xi)}=\overline{w}(\xi)$$

$$(w_0+w_1+w_2+\dots)(\overline{w}_0+\overline{w}_1+\overline{w}_2+\dots)=1$$

deg $=0$ $\quad w_0\overline{w}_0=1$

deg $=1$ $\quad w_0\overline{w}_1+w_1\overline{w}_0=0$

$\Rightarrow$ can solve for $\overline{w}_i$.

(4) If ξ is an $\mathbb{R}^n$ bundle with Euclidean metric and with (a) nowhere vanishing section. Then $w_n(\xi)=0$.

(b) k independent nowhere vanishing section. then $w_{n-k+1}(\xi)=\dots=w_n(\xi)=0$.

Fact Of all S^n only $(S^0), S^1, S^3, S^7$ are parallelizable i.e. TS^n is trivial.

Related: $\mathbb{R}$-division algebras

$$\mathbb{R} \rightsquigarrow \mathbb{C} \rightsquigarrow \underset{\text{non commu}}{\mathbb{H}} \longrightsquigarrow \mathbb{O}$$

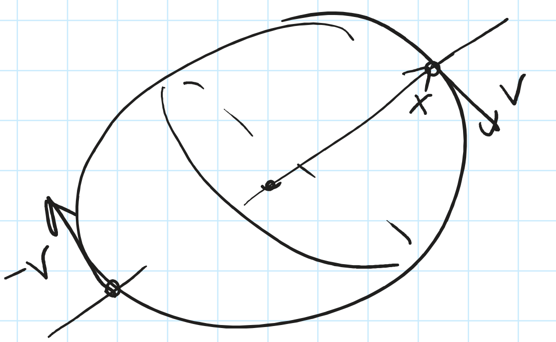
not assoc.

$$\mathbb{R}P^n = \mathbb{P}^n = S^n/\pm 1 = e^0 \cup e^1 \cup e^2 \cup \dots \cup e^n$$

$$H^*(\mathbb{R}P^n; \mathbb{F}_2) = \mathbb{F}_2[a]/a^{n+1}=0$$

Very important facts: • $w(\gamma_1^1) = 1 + a$ over $\mathbb{P}^1$

• $T\mathbb{R}P^n = \mathrm{Hom}(\gamma_n^1, (\gamma_n^1)^\perp)$



$x \cdot v = 0$

$x \cdot x = 1$