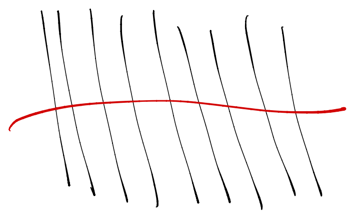
Milnor-Stasheff "Characteristic Classes" Chapter 4

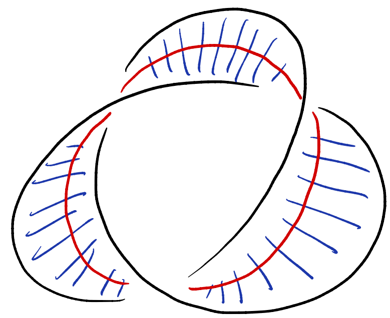
(Melissa's Personal Talk notes)

Recall

① Vector Bundles $\xi : E \to B$



locally



globally

The rank of ξ is the dimension of each fiber.

Vector bundle morphism (map): $F : E(\xi) \longrightarrow E(\eta)$ (continuous)

that is structure preserving:

$$\begin{tikzcd} E(\xi) \arrow[r, "F"] \arrow[d, "\xi"'] \arrow[dr, phantom, "\circlearrowright"] & E(\eta) \arrow[d, "\eta"] \\ B(\xi) \arrow[r, "f"'] & B(\eta) \end{tikzcd}$$

"f is covered by F"

& If $f(b) = b'$

$b \in B(\xi)$, $b' \in B(\eta)$

then the induced map on fibers

$$F_b : \xi^{-1}(b) \longrightarrow \eta^{-1}(b')$$

is a linear map

note: fiber-preserving

If v_1, v_2 are in the same fiber over $B(\xi)$, then they map to the same fiber over $B(\eta)$.

② Cohomology Rings $H^*(M;R)$ (Cohomology with coefficients)

Smooth n-manifold $M^n \rightsquigarrow$ a collection of R-modules

$H^i(M;R)$ possibly nontrivial for $0 \le i \le n$.

"Ring": Cup product $H^i(M;R) \times H^j(M;R) \xrightarrow{\smile} H^{i+j}(M;R)$

$\Rightarrow H^*(M;R)$ is a graded ring

<u>Axioms</u>: Add axioms?

Today:

Let $\xi = \xi : E \to B$ be a vector bundle.

Stiefel-Whitney classes $\{w_i(\xi)\}$, $w_i \in H^i(B(\xi); \mathbb{Z}/2)$

for $i = 0, 1, 2, \ldots$

These are invariants of vector bundles w/ base space $B(\xi)$.

(+ functorial! ie play well with morphisms of vector bundles)

<u>Axiom 1.</u> (what they are, where they live)

$$w_i(\xi) \in H^i(B(\xi); \mathbb{Z}/2) \qquad i = 0, 1, 2, \ldots$$

$$w_0(\xi) = 1 \in H^0(B(\xi); \mathbb{Z}/2)$$

$\Rightarrow$ $w_j(\xi) = 0$ when $j > \dim B(\xi)$

but <u>also</u>: $w_i(\xi) = 0$ when $i > \operatorname{rank}(\xi)$.

<u>Axiom 2</u> (Naturality)

If $f : B(\xi) \to B(\eta)$ is covered by a bundle map $\tilde{f} : \xi \to \eta$, then

$$\begin{tikzcd} E(\xi) \arrow[r, dashed, "\tilde{f}"] \arrow[d] & E(\eta) \arrow[d] \\ B(\xi) \arrow[r, "f"'] & B(\eta) \end{tikzcd}$$

$$w_i(\xi) = f^* w_i(\eta).$$ pullback

* Very strong condition $\Rightarrow$ $\{w_i\}$ don't hold "too much" info...

eg. If I understand all the $w_i(\pi)$ of a particular bundle $\pi : E \to B$, then as soon as I have a VB map $\xi \to \pi$, I know all the $w_i(\xi)$.

Axiom 3 (The Whitney Product Theorem)

$\oplus$ of VB $\longleftrightarrow$ $\smile$ of Stiefel-Whitney classes

If $B(\xi) = B(\eta)$, then

$$w_k(\xi \oplus \eta) = \sum_{i=0}^{k} w_i(\xi) \smile w_{k-i}(\eta)$$

eg. $w_1(\xi \oplus \eta) = w_1(\xi) \smile w_0(\eta) + w_0(\xi) \smile w_1(\eta) = w_1(\xi) + w_1(\eta)$

in $H^*(B; \mathbb{Z}/2)$

$w_2(\xi \oplus \eta) = w_2(\xi) + \underbrace{w_1(\xi) \smile w_1(\eta)} + w_2(\eta)$

henceforth write as $w_1(\xi) w_1(\eta)$

in the cohom ring of B.

Axiom 4 (Nontriviality)

γ_1^1 : Möbius $\longrightarrow \mathbb{RP}^1 = S^1$ tautological line bundle over $\mathbb{RP}^1$

$w_1(\gamma_1^1) \neq 0$.

(A sort of base case. We can now use the other Axioms to build the $w_i(\gamma_1^1)$ and $w_i(\xi)$ from here.)

* Assume these axioms can be simultaneously satisfied and these classes exist for all VBs.

Short Version of the Axioms

① $w_i(\xi) \in H^i(B(\xi); \mathbb{Z}/2)$

② $\exists$ VB map $\tilde{f}: \xi \to \eta$, then $\quad w_i(\xi) = f^* w_i(\eta)$ pullback

③ $w_k(\xi \oplus \eta) = \sum w_i(\xi) w_{k-i}(\eta)$

* Total Stiefel Whitney classes: $w(\xi) = w_0(\xi) + w_1(\xi) + \dots$

$$w(\xi \oplus \eta) = w(\xi)\, w(\eta)$$

ok for finite rank VBs.

④ $w_1(\gamma_1^1) \neq 0$.

* Alg. concerns? Will formalize later.

Immediate Consequences of the Four Axioms

<u>prop 1</u> $\xi \approx \eta \quad \Rightarrow \quad w_i(\xi) = w_i(\eta)$ for all η.

equivalently $w(\xi) = w(\eta)$.

Pf. use Axiom ②

<u>prop 2</u> $\varepsilon: E \to B$ is a trivial VB (ie $\mathbb{R}^n \times B$)

$\Rightarrow w_i(\varepsilon) = 0 \quad \forall\, i > 0$.

Pf.

$$\begin{tikzcd} \mathbb{R}^n \times B \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & \mathbb{R}^n \arrow[d] \\ B \arrow[r, "*"] & * \end{tikzcd}$$

$$H^*(*; \mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2 & i = 0 \\ 0 & i \neq 0 \end{cases}$$

<u>Prop 3</u> If ε trivial, then $w_i(\varepsilon \oplus \eta) = w_i(\eta)$

Pf. Use Whitney product theorem + Prop 2.

Rmk. So adding rank to VB trivially doesn't make the VB more interesting and w_i reflects this.

But.... more interesting is when you have to solve an equation with trivial bundle on other side

Compare: $\varepsilon \oplus \eta$ "$\approx$" η vs. $\xi \oplus \eta = \varepsilon$

Observation Suppose $\xi \oplus \eta = \varepsilon$.

$\Rightarrow$ $w(\xi)w(\eta) = w(\varepsilon) = \{0$ in all $H^i(B;\mathbb{Z}/2)$, $i>0$.

$(w_0(\xi) = w_0(\eta) = w_0(\varepsilon) = 1 \in H^0(B;\mathbb{Z}/2)$ — not interesting)

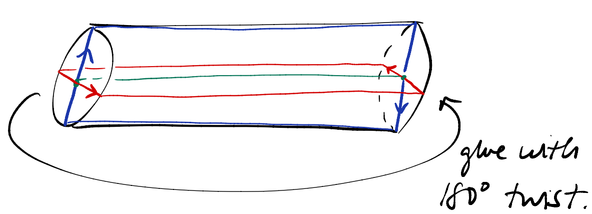
$$\begin{cases} w_1(\xi) + w_1(\eta) = 0 \\ w_2(\xi) + w_1(\xi)w_1(\eta) + w_1(\eta) = 0 \\ w_3(\xi) + w_2(\xi)w_1(\eta) + w_1(\xi)w_2(\eta) + w_3(\eta) = 0 \\ \ldots \text{ etc.} \end{cases}$$

Solve these inductively, start from the top.

$\rightsquigarrow$ can describe $w_i(\eta)$ as a polynomial in the $\{w_i(\xi)\}$.

eg. Mobius $\oplus$ Mobius$^\perp$ = $\mathbb{R}^2 \times S^1 = \varepsilon$. $(\gamma_1^1 \oplus (\gamma_1^1)^\perp = \varepsilon)$

Axiom 4 $\Rightarrow$ $w_1(\gamma_1^1) =$ "a" (since $\neq 0$)



(Exercise)

<u>Prop 4</u> If ξ is an $\mathbb{R}^n$-bundle with a Euclidean metric

with a ~~nowhere zero cross-section~~ nowhere-vanishing section (or non-)

then $w_n(\xi) = 0$.

* In fact, if ξ has k linearly indep. nonvanishing sections, then $w_{n-k+1} = w_{n-k+2}(\xi) = \cdots = w_n(\xi) = 0$.

Brief algebraic aside (see M-S for more detail)

$H^{\Pi}(B;\mathbb{Z}/2) := \{ \text{formal sums } \sum_{i=0}^{\infty} a_i \mid a_i \in H^i(B;\mathbb{Z}/2) \}$

with product operation the usual for formal power series.

$(a_0+a_1+a_2+\cdots)(b_0+b_1+b_2+\cdots) = [a_0 b_0] + [a_1 b_0 + a_0 b_1] + \cdots$

* This product is commutative (Why?)

- $a_i b_j = a_i \smile b_j = \pm b_j \smile a_i$ but we are over $\mathbb{Z}/2$!

Lemma 4.1 (Compare w/ power series)

$\{ \sum a_i \mid a_0 = 1 \}$ is the group of units $(H^{\Pi}(B;\mathbb{Z}/2))^{\times}$.

Pf. $w = \sum a_i = 1 + w_1 + w_2 + \cdots$

Construct the inverse $\bar{w} = 1 + \bar{w}_1 + \bar{w}_2 + \cdots$

by solving $w\bar{w} = 1$.

Book contains the formula.

Use the power series expansion as usual:

$\bar{w} = [1 + \underbrace{(w_1 + w_2 + \cdots)}_{X}]^{-1} = [1-(-X)]^{-1} = \sum_{k=0}^{\infty} (-X)^k$

$= 1 - (w_1+w_2+w_3+\cdots) + (w_1+w_2+w_3+\cdots)^2 - (w_1+w_2+w_3+\cdots)^3 + \cdots$

Reorganize by homological grading/degree
ie which H^i they're actually in

$= 1 - w_1 + (w_1^2 - w_2) + (-w_1^3 + 2w_1 w_2 - w_3) + \cdots$

$\Rightarrow$ hence there is a formula.

Since $w_0(\xi) = 1 \ \forall \xi$, we always have inverses!

$$w(\xi \oplus \eta) = w(\xi) w(\eta) \quad \Rightarrow \quad w(\eta) = \bar{w}(\xi) w(\xi \oplus \eta)$$

So if $\xi \oplus \eta = \varepsilon$, then

$$w(\xi) w(\eta) = 1 \quad \Rightarrow \quad w(\eta) = \bar{w}(\xi)$$

eg. $TM^n \oplus NM^n = \mathbb{R}^n \times M^n$ $\quad$ (τ_M, ν_M in book.)

so tangent & normal VB are algebraically dual from the point of view of Stiefel-Whitney char classes.

<u>notation</u> $w(M) =$ total S-W class of TM.
denoted τ_M in M-S.

Applications.

$\mathbb{R}P^n$ etc

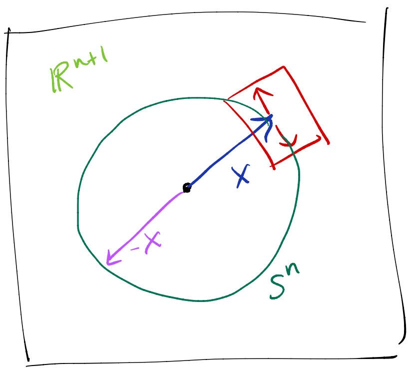
S^1, S^3, S^7 etc & Div. Algebras / parallelizability

cobordism classes

11

More symbolically: $E(\gamma^\perp) = (\{\pm x\}, v\} \in P^n \times \mathbb{R}^{n+1}$

where $x \perp v$.



$$\underbrace{(1-a)}_{\substack{w(\gamma_n^1) \\ (=1+a) \\ \text{over } \mathbb{F}_2}}\underbrace{(1+a+\dots+a^n)}_{\substack{\bar{w}(\gamma_n^1) \\ = w((\gamma_n^1)^\perp)}} = 1 + \cancel{a^{n+1}}^{\,0} = 1$$

//

Eg 4. $TP^n \cong \mathrm{Hom}(\gamma_n^1, (\gamma_n^1)^\perp)$.

on fibers: $\ell : L \longrightarrow L^\perp$

Main idea here: You need to relate linear map w/ the

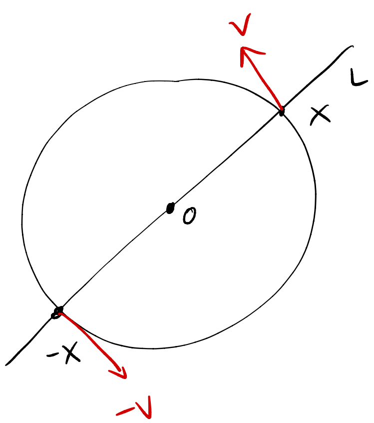
Main idea here: $T_{\{\pm x\}} P^n \longrightarrow$ "project to orthogonal complement" linear map.

$\{(x,v), (-x,-v)\}$
s.t. $x \cdot x = 1$
$x \cdot v = 0$.

$\ell : L \longrightarrow L^\perp$

$x \longmapsto v$

spans whole line.



Below here: did not get to in class.

Stiefel-Whitney classes of $T\mathbb{P}^n$:

Thm $T\mathbb{P}^1 \oplus \varepsilon^1_{\mathbb{P}^1} \cong \underbrace{\gamma_n^1 \oplus \cdots \oplus \gamma_n^1}_{n+1}$

triv. line bundle over $\mathbb{P}^1$

$$\Rightarrow \quad w(T\mathbb{P}^n) = (1+a)^{n+1} = \sum_{k=0}^{n+1} \binom{n+1}{k} a^k$$

$\boxed{\text{Pf of } \Rightarrow}$ Whitney product theorem (Axiom 3).

$\boxed{\text{Pf of } \circledast}$

$$T\mathbb{P}^n \oplus \varepsilon^1 \cong \mathrm{Hom}(\gamma_n^1, \gamma^\perp) \oplus \mathrm{Hom}(\gamma_n^1, \gamma_n^1)$$

$$\cong \mathrm{Hom}(\gamma_n^1, \underbrace{\gamma^\perp \oplus \gamma_n^1}_{\varepsilon^{n+1}})$$

$$\varepsilon^{n+1} = \underbrace{\varepsilon^1 \oplus \cdots \oplus \varepsilon^1}_{n+1}$$

$$\cong \bigoplus^{n+1} \mathrm{Hom}(\gamma_n^1, \varepsilon^1).$$

Parallelizable spheres & real division algebras

Only $(S^0,)\, S^1, S^3, S^7$ are parallelizable

$\rightsquigarrow \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ $\mathbb{R}$-division algebras.

Hamiltonian quaternions

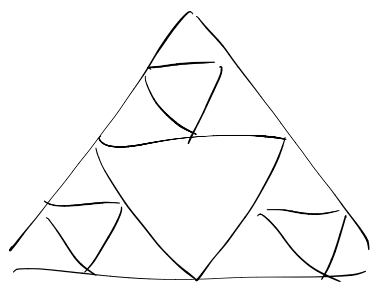
(ie $\mathbb{R}$ algebras $\approx$ fields but • not necessary commutative

eg. in $\mathbb{H}$, $ij = -ji = k$.)

or assoc. (in $\mathbb{O}$).

"$w(\mathbb{P}^n)$" $= w(T\mathbb{P}^n)$: coeffs by Pascal Δ mod 2 (Sierpinski gasket)

(& right row is coeff. of $a^{n+1} = 0$.)



Viewing Sierpinski Gasket, $w(\mathbb{P}^n) \equiv 1$ iff $n+1 = 2^r$

$(1+a)^{2^r} = 1 + a^{2^r}$ (Freshman's binomial)

(Can also easily prove converse)

thm 4.7 (Stiefel) Suppose $\exists$ bilinear product operation

$p: \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$

w/o zero divisors (ie like field, ~~group of~~ units is $A - \{0\}$.)

Then $\mathbb{P}^{n-1}$ is parallelizable and hence $n = 2^r$.

Proof. (of 4.7).

Go over yourself!

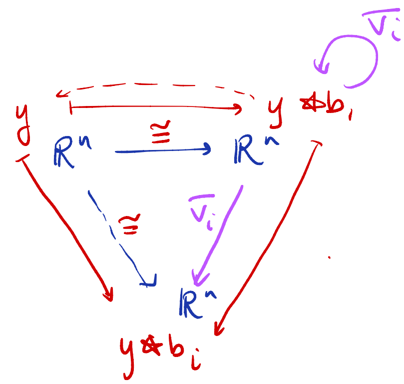
Proof of 4.7 $\quad p: \mathbb{R}^n \times \mathbb{R}^n \xrightarrow{*} \mathbb{R}^n$

$*$ not nec commu or even assoc!

(but its bilinear!)

$\mathbb{R}^n = \text{span}\{b_1, \dots, b_n\}$.

$y \mapsto y * b_1$ is an Vect isom $\mathbb{R}^n \to \mathbb{R}^n$.



Define $v_i(y * b_1) = y * b_i$

$\rightsquigarrow v_1(x), \dots, v_n(x)$

$\forall x \neq 0$, $\{v_1(x), \dots, v_n(x)\}$ are linearly indep.

& $v_1(x) = x$.

$\Rightarrow v_2, \dots, v_n \rightsquigarrow n-1$ linearly indep nowhere-vanishing sections of

$\underset{v_i}{TP^{n-1}} \cong \underset{\bar{v}_i}{\text{Hom}(\gamma^1_{n-1}, \gamma^\perp)}$: as defn below:

For $x \in L$, $\bar{v}_i(x) =$ img of $v_i(x)$ under orth. proj: $\mathbb{R}^n \to L^\perp$

So $\bar{v}_1 = 0$, but $\bar{v}_2, \dots, \bar{v}_n$ are everywhere lin. indep.

$\Rightarrow TP^{n-1}$ is trivial bundle. //

Also fun:

① Q. When can P^{2^r} be immersed in $\mathbb{R}^{2^r+k}$?

A: $k \geq 2^r - 1$.

② Stiefel Whitney #s and (unoriented) cobordism classes.

$M_1 \sim M_2$ iff $M_1 \cup M_2 = \partial$ smooth compact $(n+1)$-dim mfld.

Fact. 2 smooth closed n manifolds are in same cobord class

iff same SW #s.